

# MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS

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**ABSTRACT.** We give a definition of monoidal categorifications of quantum cluster algebras and provide a criterion for a monoidal category of finite-dimensional graded  $R$ -modules to become a monoidal categorification of a quantum cluster algebra, where  $R$  is a symmetric Khovanov-Lauda-Rouquier algebra. Roughly speaking, this criterion asserts that a quantum monoidal seed can be mutated successively in all the directions once the first-step mutations are possible. In the course of the study, we also give a proof of a conjecture of Leclerc on the product of upper global basis elements.

## INTRODUCTION

The purpose of this paper is to give a definition of a monoidal categorification of a quantum cluster algebra and to provide a criterion for a monoidal category to be a monoidal categorification.

The notion of cluster algebras was introduced by Fomin and Zelevinsky in [5] for studying total positivity and upper global bases. Since their introduction, connections and applications have been discovered in various fields of mathematics including representation theory, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

A cluster algebra is a  $\mathbb{Z}$ -subalgebra of a rational function field given by a set of generators, called the *cluster variables*. These generators are grouped into overlapping subsets, called *clusters*, and the clusters are defined inductively by a procedure called *mutation* from the *initial cluster*  $\{X_i\}_{1 \leq i \leq r}$ , which is controlled by an exchange matrix  $\tilde{B}$ . We call a monomial of cluster variables in one cluster a *cluster monomial*.

Fomin and Zelevinsky proved that every cluster variable is a Laurent polynomial of the initial cluster  $\{X_i\}_{1 \leq i \leq r}$  and they conjectured that this Laurent polynomial has positive coefficients ([5]). This *positivity conjecture* was proved by Lee and Schiffler in

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the *skew-symmetric* cluster algebra case in [19]. The *linearly independence conjecture* on cluster monomials was proved in the skew-symmetric cluster algebra case in [3].

The notion of quantum cluster algebras, introduced by Berenstein and Zelevinsky in [2], can be considered as a  $q$ -analogue of cluster algebras. The commutation relation among the cluster variables is determined by a skew-symmetric matrix  $L$ . As in the cluster algebra case, every cluster variable belongs to  $\mathbb{Z}[q^{\pm 1/2}][X_i^{\pm 1}]_{1 \leq i \leq r}$  ([2]), and is expected to be an element of  $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}][X_i^{\pm 1}]_{1 \leq i \leq r}$ , which is referred to as the *quantum positivity conjecture* (cf. [4, Conjecture 4.7]). In [16], Kimura and Qin proved the quantum positivity conjecture for quantum cluster algebras containing *acyclic* seed and specific coefficients.

In a series of papers [6, 7, 8], Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra  $A_q(\mathfrak{n}(w))$ , associated with a symmetric quantum group  $U_q(\mathfrak{g})$  and its Weyl group element  $w$ , has a skew-symmetric quantum cluster algebra structure whose initial cluster consists of *quantum minors*. In [15], Kimura proved that  $A_q(\mathfrak{n}(w))$  is *compatible* with the upper global basis  $\mathbf{B}^{\text{up}}$  of  $A_q(\mathfrak{n})$ ; i.e., the set  $\mathbf{B}^{\text{up}}(w) := A_q(\mathfrak{n}(w)) \cap \mathbf{B}^{\text{up}}$  is a basis of  $A_q(\mathfrak{n}(w))$ . Thus, with a result of [3], one can expect that every cluster monomial of  $A_q(\mathfrak{n}(w))$  is contained in the upper global basis  $\mathbf{B}^{\text{up}}(w)$ , which is named *the quantization conjecture* by Kimura ([15]).

In [9], Hernandez and Leclerc introduced the notion of *a monoidal categorification of a cluster algebra*. We say that a simple object  $S$  of a monoidal category  $\mathcal{C}$  is *real* if  $S \otimes S$  is simple. We say that a simple object  $S$  is *prime* if there exists no non-trivial factorization  $S \simeq S_1 \otimes S_2$ . They say that  $\mathcal{C}$  is a monoidal categorification of a cluster algebra  $A$  if the Grothendieck ring of  $\mathcal{C}$  is isomorphic to  $A$  and if

- (M1) the cluster monomials of  $A$  are the classes of real simple objects of  $\mathcal{C}$ ,
- (M2) the cluster variables of  $A$  are the classes of real simple prime objects of  $\mathcal{C}$ .

(Note that the above version is weaker than the original definition of the monoidal categorification in [9].) They proved that certain categories of modules over symmetric quantum affine algebras  $U'_q(\mathfrak{g})$  give monoidal categorifications of cluster algebras. Nakajima extended it to the cases of the cluster algebras of type  $A, D, E$  ([21]) (see also [10]).

Once a cluster algebra  $A$  has a monoidal categorification, the positivity of cluster variables of  $A$  and the linear independency of cluster monomials of  $A$  follow (see [9, Proposition 2.2]).

In this paper, we will refine their notion of monoidal categorifications including the quantum cluster algebra case.

The Khovanov-Lauda-Rouquier (or simply KLR) algebras, introduced by Khovanov-Lauda [13, 14] and Rouquier [22] independently, are a family of  $\mathbb{Z}$ -graded algebras which categorifies the negative half  $U_q^-(\mathfrak{g})$  of a *symmetrizable* quantum group  $U_q(\mathfrak{g})$ . More

precisely, there exists a family of algebras  $\{R(-\beta)\}_{\beta \in \mathbf{Q}^-}$  such that the Grothendieck ring of  $R\text{-gmod} := \bigoplus_{\beta \in \mathbf{Q}^-} R(-\beta)\text{-gmod}$ , the direct sum of the categories of finite-dimensional graded  $R(-\beta)$ -modules, is isomorphic to the integral form  $A_q(\mathbf{n})_{\mathbb{Z}[q^{\pm 1}]}$  of  $A_q(\mathbf{n}) \simeq U_q^-(\mathfrak{g})$ . Here the tensor functor  $\otimes$  of the monoidal category  $R\text{-gmod}$  is given by the convolution product  $\circ$ , and the action of  $q$  is given by the grading shift functor. In [24, 23], Varagnolo-Vasserot and Rouquier proved that the upper global basis  $\mathbf{B}^{\text{up}}$  of  $A_q(\mathbf{n})$  corresponds to the set of the classes of all *self-dual* simple modules of  $R\text{-gmod}$  under the assumption that  $R$  is associated with a *symmetric* quantum group  $U_q(\mathfrak{g})$ .

Combining works of [8, 15, 24], the quantum unipotent coordinate algebra  $A_q(\mathbf{n}(w))$  associated with a symmetric quantum group  $U_q(\mathfrak{g})$  and a Weyl group element  $w$  is isomorphic to the Grothendieck group of the monoidal abelian full subcategory  $\mathcal{C}_w$  of  $R\text{-gmod}$  satisfying the following properties: (i)  $\mathcal{C}_w$  is stable under extensions and grading shift functor, (ii) the composition factors of  $M \in \mathcal{C}_w$  are contained in  $\mathbf{B}^{\text{up}}(w)$ . However it is not evident the conditions (M1) and (M2) are satisfied. The purpose of this paper is provide a theoretical background in order to prove (M1) and (M2). In the forthcoming paper, we prove (M1) and (M2) as an application of the results of the present paper.

In this paper, we first continue the work of [12] about the convolution products, heads and socles of graded modules over symmetric KLR algebras. One of the main results in [12] is that the convolution product  $M \circ N$  of a real simple  $R(\beta)$ -module  $M$  and a simple  $R(\gamma)$ -modules  $N$  has a unique simple quotient and a unique simple submodule. Moreover, if  $M \circ N \simeq N \circ M$  up to a grading shift,  $M \circ N$  is simple. In such a case we say that  $M$  and  $N$  *commute*. The main tool of [12] was an R-matrix  $\mathbf{r}_{M,N}$ , constructed in [11], which is a homogeneous homomorphism from  $M \circ N$  to  $N \circ M$  of degree  $\Lambda(M, N)$ . We define the integers

$$\tilde{\Lambda}(M, N) := \frac{1}{2}(\Lambda(M, N) + (\beta, \gamma)), \quad \mathfrak{d}(M, N) := \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M))$$

and study the representation theoretic meaning of the integers  $\Lambda(M, N)$ ,  $\tilde{\Lambda}(M, N)$  and  $\mathfrak{d}(M, N)$ .

We then prove Leclerc's first conjecture ([18]) on the multiplicative structure of elements in  $\mathbf{B}^{\text{up}}$  when the generalized Cartan matrix is symmetric (Theorem 3.1 and Theorem 3.6). Theorem 3.6 is due to McNamara ([20, Lemma 7.5]) and the authors thank him for informing us his result.

We say that  $b \in \mathbf{B}^{\text{up}}$  is *real* if  $b^2 \in q^{\mathbb{Z}} \mathbf{B}^{\text{up}} := \bigsqcup_{n \in \mathbb{Z}} q^n \mathbf{B}^{\text{up}}$ .

**Conjecture** ([18, Conjecture 1]). *Let  $b_1$  and  $b_2$  be elements in  $\mathbf{B}^{\text{up}}$  such that  $b_1$  is real and  $b_1 b_2 \notin q^{\mathbb{Z}} \mathbf{B}^{\text{up}}$ . Then the expansion of  $b_1 b_2$  with respect to  $\mathbf{B}^{\text{up}}$  is of the form*

$$b_1 b_2 = q^m b' + q^s b'' + \sum_{c \neq b', b''} \gamma_{b_1, b_2}^c(q) c,$$

where  $b' \neq b''$ ,  $m, s \in \mathbb{Z}$ ,  $m < s$ ,  $\gamma_{b_1, b_2}^c(q) \in \mathbb{Z}[q^{\pm 1}]$ , and for all  $c \in \mathbf{B}^{\text{up}}$  such that  $\gamma_{b_1, b_2}^c(q) \neq 0$

$$\gamma_{b_1, b_2}^c(q) \in q^{m+1}\mathbb{Z}[q] \cap q^{s-1}\mathbb{Z}[q^{-1}].$$

More precisely, we prove that  $q^m b'$  and  $q^s b''$  correspond to the simple head and the simple socle of  $M \circ N$ , respectively, when  $b_1$  corresponds to a real simple module  $M$  and  $b_2$  corresponds to a simple module  $N$ .

In the second part of this paper, we provide an algebraic framework for monoidal categorifications of cluster algebras and quantum cluster algebras. Let us focus on the quantum cluster algebra case in this introduction.

Let  $J$  be a finite index set with a decomposition  $J_{\text{ex}} \sqcup J_{\text{fr}}$  and let  $\mathbf{Q}$  be a free abelian group with a symmetric bilinear form  $(\ , \ )$  such that  $(\beta, \beta) \in 2\mathbb{Z}$  for all  $\beta \in \mathbf{Q}$ . Let  $\mathcal{C}$  be an abelian monoidal category with a *grading shift functor*  $q$  (see (5.4)) and a decomposition  $\mathcal{C} = \bigoplus_{\beta \in \mathbf{Q}} \mathcal{C}_{\beta}$ . The quadruple  $\mathcal{S} := (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  is called a *quantum monoidal seed* when it consists of (i) an integer-valued  $J \times J_{\text{ex}}$ -matrix  $\tilde{B}$  with the skew-symmetric principal part, (ii) an integer-valued skew-symmetric  $J \times J$ -matrix  $L = (\lambda_{ij})_{i, j \in J}$  such that  $(L, \tilde{B})$  is *compatible* with the integer 2, (iii) the set of real simple modules  $\{M_i\}_{i \in J}$  in  $\mathcal{C}$  such that  $M_i \otimes M_j \simeq q^{\lambda_{ij}} M_j M_i$  and  $M_{i_1} \otimes \cdots \otimes M_{i_{\ell}}$  is simple for any  $i, j \in J$  and  $(i_1, \dots, i_{\ell}) \in J^{\ell}$ , (iv)  $D = (\text{wt}(M_i))_{i \in J} \in \mathbf{Q}^J$ , and it satisfies certain conditions (Definition 5.4). Here we set  $\text{wt}(M) = \beta$  for  $M \in \mathcal{C}_{\beta}$ .

For  $k \in J_{\text{ex}}$ , we say that  $\mathcal{S}$  *admits a mutation in direction  $k$*  if there exists a simple object  $M'_k$  such that there exist exact sequences in  $\mathcal{C}$

$$\begin{aligned} 0 \longrightarrow q \bigotimes_{b_{ik} > 0} M_i^{\odot b_{ik}} &\longrightarrow q^{m_k} M_k \otimes M'_k \longrightarrow \bigotimes_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \longrightarrow 0, \\ 0 \longrightarrow q \bigotimes_{b_{ik} < 0} M_i^{\odot (-b_{ik})} &\longrightarrow q^{m'_k} M'_k \otimes M_k \longrightarrow \bigotimes_{b_{ik} > 0} M_i^{\odot b_{ik}} \longrightarrow 0, \end{aligned}$$

and its *mutation*  $\mu_k(\mathcal{S}) := (\{\mu_k(M)_i\}_{i \in J}, \mu_k(L), \mu_k(\tilde{B}), \mu_k(D))$  is again a quantum monoidal seed. Here  $\bigotimes$  denotes a tensor product with some grading shift (see (5.6)), and  $m_k, m'_k$  are some integers determined by  $(L, \tilde{B}, D)$  (see Lemma 5.5 and Definition 5.6 for more precise definition).

We say that  $\mathcal{C}$  is a *monoidal categorification* of a skew-symmetric quantum cluster algebra  $A$  if (i) the Grothendieck ring of  $\mathcal{C}$  is isomorphic to  $A$ , (ii) there exists a quantum monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  in  $\mathcal{C}$  such that, for some  $m_i \in \frac{1}{2}\mathbb{Z}$  ( $i \in J$ ),  $[\mathcal{S}] := (\{q^{m_i}[M_i]\}_{i \in J}, L, \tilde{B})$  is an initial quantum seed of  $A$  (iii)  $\mathcal{S}$  admits successive mutations in all directions.

The existence of monoidal category  $\mathcal{C}$  which provides a monoidal categorification of quantum cluster algebra  $A$  implies the followings:

- (QM1) Every quantum cluster monomial corresponds to the isomorphism class of a real simple object of  $\mathcal{C}$ . In particular, the set of quantum cluster monomials is  $\mathbb{Z}[q^{\pm 1/2}]$ -linearly independent.
- (QM2) The quantum positivity conjecture holds for  $A$ .

In the last section of this paper, we study the case when the category  $\mathcal{C}$  is a full subcategory of  $R\text{-gmod}$  which is stable under taking convolution products, subquotients, extensions and grading shifts. Then the category has a natural decomposition  $\mathcal{C} = \bigoplus_{\beta \in Q^-} \mathcal{C}_\beta$  for  $\mathcal{C}_\beta = \mathcal{C} \cap R(-\beta)\text{-gmod}$ .

We say that a pair  $(\{M_i\}_{i \in J}, \tilde{B})$  is *admissible* if (i)  $\{M_i\}_{i \in J}$  is a family of real simple *self-dual* objects in  $\mathcal{C}$  which commute with each other, (ii)  $\tilde{B}$  is an integer-valued  $J \times J_{\text{ex}}$ -matrix with skew-symmetric part, (iii) for any  $k \in J_{\text{ex}}$ , there exists a self-dual simple object  $M'_k$  in  $\mathcal{C}$  such that there is an exact sequence in  $\mathcal{C}$

$$0 \rightarrow q \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \rightarrow q^{\tilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \rightarrow \bigodot_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \rightarrow 0,$$

and  $M'_k$  commutes with  $M_i$  for any  $i \neq k$  (Definition 6.1).

The main theorem of our paper is the following (Theorem 6.3, Corollary 6.4):

**Main Theorem.** Set  $\Lambda := (\Lambda(M_i, M_j))_{i, j \in J}$ , and let  $\mathcal{S} = (\{M_i\}_{i \in J}, -\Lambda, \tilde{B}, D)$  be a quantum monoidal seed in  $\mathcal{C}$ . We assume further the following condition:

- The Grothendieck ring of  $\mathcal{C}$  is isomorphic to the quantum cluster algebra  $A$  associated to the initial quantum seed  $[\mathcal{S}] := (\{q^{-\frac{1}{4}(d_i, d_i)}[M_i]\}_{i \in J}, -\Lambda, \tilde{B})$  with  $d_i := \text{wt}(M_i)$ .

If the pair  $(\{M_i\}_{i \in J}, \tilde{B})$  is admissible, then the category  $\mathcal{C}$  is a monoidal categorification of the quantum cluster algebra  $A$ .

When the base field of the symmetric KLR algebra is of characteristic 0, Main Theorem, along with Theorem 1.5 due to [24, 23], implies the quantization conjecture:

- (QM3) The set of cluster monomials of  $A$  is contained in the upper global basis  $\mathbf{B}^{\text{up}}$ .

In the forthcoming paper, we will prove the existence of admissible quantum monoidal seeds in  $\mathcal{C}_w$ . Hence, Main Theorem shows that the category  $\mathcal{C}_w$  provides a monoidal categorification of the quantum cluster algebra  $A_q(\mathbf{n}(w))$ , and (QM3) holds for  $A_q(\mathbf{n}(w))$ .

The paper is organized as follows. In Section 1, we briefly review some of basic materials on quantum groups and KLR algebras. In Section 2, we continue the study in [12] about convolution products, heads and socles of  $R$ -modules. In Section 3, we prove the first conjecture of Leclerc in [18]. In Section 4, we recall the definition of

quantum cluster algebras. In Section 5, we give the definitions of a monoidal seed, a quantum monoidal seed, a monoidal categorification of a cluster algebra and a monoidal categorification of a quantum cluster algebra. In Section 6, we prove Main Theorem by using the results of previous sections.

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## 1. QUANTUM GROUPS AND KLR ALGEBRAS

**1.1. Quantum groups and upper global bases.** Let  $I$  be an index set. A *Cartan datum* is a quintuple  $(A, P, \Pi, P^\vee, \Pi^\vee)$  consisting of

- (a) an integer-valued matrix  $A = (a_{ij})_{i,j \in I}$ , called the *symmetrizable generalized Cartan matrix*, which satisfies
  - (1)  $a_{ii} = 2$  ( $i \in I$ ),
  - (2)  $a_{ij} \leq 0$  ( $i \neq j$ ),
  - (3) there exists a diagonal matrix  $D = \text{diag}(s_i \mid i \in I)$  such that  $DA$  is symmetric, and  $s_i$  are positive integers.
- (b) a free abelian group  $P$ , called the *weight lattice*,
- (c)  $\Pi = \{\alpha_i \in P \mid i \in I\}$ , called the set of *simple roots*,
- (d)  $P^\vee := \text{Hom}(P, \mathbb{Z})$ , called the *co-weight lattice*,
- (e)  $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ , called the set of *simple coroots*, satisfying the following properties:
  - (1)  $\langle h_i, \alpha_j \rangle = a_{ij}$  for all  $i, j \in I$ ,
  - (2)  $\Pi$  is linearly independent,
  - (3) for each  $i \in I$ , there exists  $\Lambda_i \in P$  such that  $\langle h_j, \Lambda_i \rangle = \delta_{ij}$  for all  $j \in I$ .

We call  $\Lambda_i$  the *fundamental weights*.

The free abelian group  $\mathbb{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  is called the *root lattice*. Set  $\mathbb{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i \subset \mathbb{Q}$  and  $\mathbb{Q}^- = \sum_{i \in I} \mathbb{Z}_{\leq 0}\alpha_i \subset \mathbb{Q}$ . For  $\beta = \sum_{i \in I} m_i \alpha_i \in \mathbb{Q}$ , we set  $|\beta| = \sum_{i \in I} |m_i|$ .

Set  $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$ . Then there exists a symmetric bilinear form  $(\ , \ )$  on  $\mathfrak{h}^*$  satisfying

$$(\alpha_i, \alpha_j) = s_i a_{ij} \quad (i, j \in I) \quad \text{and} \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Let  $q$  be an indeterminate. For each  $i \in I$ , set  $q_i = q^{s_i}$ .

**Definition 1.1.** The quantum group associated with a Cartan datum  $(A, P, \Pi, P^\vee, \Pi^\vee)$  is the algebra  $U_q(\mathfrak{g})$  over  $\mathbb{Q}(q)$  generated by  $e_i, f_i$  ( $i \in I$ ) and  $q^h$  ( $h \in P^\vee$ ) satisfying the

following relations:

$$\begin{aligned}
q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P, \\
q^h e_i q^{-h} &= q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \quad \text{for } h \in P^\vee, i \in I, \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = q^{s_i h_i}, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r &= 0 \quad \text{if } i \neq j, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r &= 0 \quad \text{if } i \neq j.
\end{aligned}$$

Here, we set  $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ ,  $[n]_i! = \prod_{k=1}^n [k]_i$  and  $\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}$  for  $i \in I$  and  $m, n \in \mathbb{Z}_{\geq 0}$  such that  $m \geq n$ .

Let  $U_q^+(\mathfrak{g})$  (resp.  $U_q^-(\mathfrak{g})$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $e_i$ 's (resp.  $f_i$ 's), and let  $U_q^0(\mathfrak{g})$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $q^h$  ( $h \in P^\vee$ ). Then we have the *triangular decomposition*

$$U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),$$

and the *weight space decomposition*

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}} U_q(\mathfrak{g})_\beta,$$

where  $U_q(\mathfrak{g})_\beta := \{x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \beta \rangle} x \text{ for any } h \in P\}$ .

For any  $i \in I$ , there exists a unique  $\mathbb{Q}(q)$ -linear endomorphism  $e'_i$  of  $U_q^-(\mathfrak{g})$  such that

$$e'_i(f_j) = \delta_{i,j} \quad (j \in I), \quad e'_i(xy) = (e'_i x)y + q_i^{\langle h_i, \beta \rangle} x(e'_i y) \quad (x \in U_q^-(\mathfrak{g})_\beta, y \in U_q^-(\mathfrak{g})).$$

Recall that there exists a unique non-degenerate symmetric bilinear form  $(\ , \ )_-$  on  $U_q^-(\mathfrak{g})$  such that

$$(\mathbf{1}, \mathbf{1}) = 1, \quad (f_i u, v)_- = (u, e'_i v)_- \quad \text{for } i \in I, u, v \in U_q^-(\mathfrak{g}).$$

Let  $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$  and set

$$e_i^{(n)} = e_i^n / [n]_i!, \quad f_i^{(n)} = f_i^n / [n]_i! \quad (n \in \mathbb{Z}_{\geq 0}).$$

Let  $U_{\mathbf{A}}^-(\mathfrak{g})$  be the  $\mathbf{A}$ -subalgebra of  $U_q^-(\mathfrak{g})$  generated by  $f_i^{(n)}$  for  $i \in I, n \in \mathbb{Z}_{\geq 0}$ . It is called the *integral form* of  $U_q^-(\mathfrak{g})$ . The *dual integral form*  $U_{\mathbf{A}}^-(\mathfrak{g})^\vee$  of  $U_q^-(\mathfrak{g})$  is defined by

$$U_{\mathbf{A}}^-(\mathfrak{g})^\vee := \{x \in U_q^-(\mathfrak{g}) \mid (x, U_{\mathbf{A}}^-(\mathfrak{g}))_- \subset \mathbf{A}\}.$$

Then  $U_{\mathbf{A}}^-(\mathfrak{g})^\vee$  has an  $\mathbf{A}$ -algebra structure as a subalgebra of  $U_q^-(\mathfrak{g})$ .



For each  $i \in I$ , any element  $x \in U_q^-(\mathfrak{g})$  can be written uniquely as

$$x = \sum_{n \geq 0} f_i^{(n)} x_n \quad \text{with } x_n \in \text{Ker}(e'_i).$$

We define the *Kashiwara operators*  $\tilde{e}_i, \tilde{f}_i$  on  $U_q^-(\mathfrak{g})$  by

$$\tilde{e}_i x = \sum_{n \geq 1} f_i^{(n-1)} x_n, \quad \tilde{f}_i x = \sum_{n \geq 0} f_i^{(n+1)} x_n,$$

and set

$$\begin{aligned} L(\infty) &= \sum_{\ell \geq 0, i_1, \dots, i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot \mathbf{1} \subset U_q^-(\mathfrak{g}), \quad \overline{L(\infty)} = \{\overline{x} \mid x \in L(\infty)\}, \\ B(\infty) &= \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot \mathbf{1} \mod qL(\infty) \mid \ell \geq 0, i_1, \dots, i_\ell \in I \right\} \subset L(\infty)/qL(\infty), \end{aligned}$$

where  $\mathbf{A}_0 = \{g \in \mathbb{Q}(q) \mid g \text{ is regular at } q = 0\}$  and  $- : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$  is the  $\mathbb{Q}$ -algebra involution given by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h}, \quad \text{and } \overline{q} = q^{-1}.$$

The set  $B(\infty)$  is a  $\mathbb{Q}$ -basis of  $L(\infty)/qL(\infty)$  and the natural map

$$L(\infty) \cap \overline{L(\infty)} \cap U_{\mathbf{A}}^-(\mathfrak{g}) \rightarrow L(\infty)/qL(\infty)$$

is a  $\mathbb{Q}$ -linear isomorphism. Let us denote the inverse of the above isomorphism by  $G^{\text{low}}$ . Then the set

$$\mathbf{B}^{\text{low}} := \{G^{\text{low}}(b) \mid b \in B(\infty)\}$$

forms an  $\mathbf{A}$ -basis of  $U_{\mathbf{A}}^-(\mathfrak{g})$  and is called the *lower global basis* of  $U_q^-(\mathfrak{g})$ .

For each  $b \in B(\infty)$ , define  $G^{\text{up}}(b) \in U_q^-(\mathfrak{g})$  as the element satisfying

$$(G^{\text{up}}(b), G^{\text{low}}(b'))_- = \delta_{b,b'}$$

for  $b' \in B(\infty)$ . The set

$$\mathbf{B}^{\text{up}} := \{G^{\text{up}}(b) \mid b \in B(\infty)\}$$

forms an  $\mathbf{A}$ -basis of  $U_{\mathbf{A}}^-(\mathfrak{g})^\vee$  and is called the *upper global basis* of  $U_q^-(\mathfrak{g})$ .

## 1.2. KLR algebras.

Now we recall the definition of Khovanov-Lauda-Rouquier algebra or quiver Hecke algebras (hereafter, we abbreviate it as KLR algebras) associated with a given Cartan datum  $(A, P, \Pi, P^\vee, \Pi^\vee)$ .

Let  $\mathbf{k}$  be a base field. For  $i, j \in I$  such that  $i \neq j$ , set

$$S_{i,j} = \{(p, q) \in \mathbb{Z}_{\geq 0}^2 \mid (\alpha_i, \alpha_i)p + (\alpha_j, \alpha_j)q = -2(\alpha_i, \alpha_j)\}.$$



Let us take a family of polynomials  $(Q_{ij})_{i,j \in I}$  in  $\mathbf{k}[u, v]$  which are of the form

$$(1.1) \quad Q_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \sum_{(p,q) \in S_{i,j}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j \end{cases}$$

with  $t_{i,j;p,q} \in \mathbf{k}$  such that  $Q_{i,j}(u, v) = Q_{j,i}(v, u)$  and  $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$ .

We denote by  $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$  the symmetric group on  $n$  letters, where  $s_i := (i, i+1)$  is the transposition of  $i$  and  $i+1$ . Then  $\mathfrak{S}_n$  acts on  $I^n$  by place permutations.

For  $n \in \mathbb{Z}_{\geq 0}$  and  $\beta \in \mathbf{Q}^+$  such that  $|\beta| = n$ , we set

$$I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \dots + \alpha_{\nu_n} = \beta\}.$$

**Definition 1.2.** For  $\beta \in \mathbf{Q}^+$  with  $|\beta| = n$ , the Khovanov-Lauda-Rouquier algebra  $R(\beta)$  at  $\beta$  associated with a Cartan datum  $(A, P, \Pi, P^\vee, \Pi^\vee)$  and a matrix  $(Q_{ij})_{i,j \in I}$  is the algebra over  $\mathbf{k}$  generated by the elements  $\{e(\nu)\}_{\nu \in I^\beta}$ ,  $\{x_k\}_{1 \leq k \leq n}$ ,  $\{\tau_m\}_{1 \leq m \leq n-1}$  satisfying the following defining relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu,\nu'} e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \\ x_k x_m &= x_m x_k, \quad x_k e(\nu) = e(\nu) x_k, \\ \tau_m e(\nu) &= e(s_m(\nu)) \tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \quad \text{if } |k - m| > 1, \\ \tau_k^2 e(\nu) &= Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\ (\tau_k x_m - x_{s_k(m)} \tau_k) e(\nu) &= \begin{cases} -e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) &= \begin{cases} \frac{Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The above relations are homogeneous provided with

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}),$$

and hence  $R(\beta)$  is a  $\mathbb{Z}$ -graded algebra.

For a graded  $R(\beta)$ -module  $M = \bigoplus_{k \in \mathbb{Z}} M_k$ , we define  $qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k$ , where

$$(qM)_k = M_{k-1} \quad (k \in \mathbb{Z}).$$

We call  $q$  the *grading shift functor* on the category of graded  $R(\beta)$ -modules.

If  $M$  is an  $R(\beta)$ -module, then we set  $\text{wt}(M) = -\beta \in \mathbf{Q}^-$  and call it the *weight* of  $M$ .

We denote by  $R(\beta)\text{-Mod}$  the category of  $R(\beta)$ -modules, and by  $R(\beta)\text{-mod}$  the full subcategory of  $R(\beta)\text{-Mod}$  consisting of modules  $M$  such that  $M$  are finite-dimensional over  $\mathbf{k}$ , and the actions of the  $x_k$ 's on  $M$  are nilpotent.

Similarly, we denote by  $R(\beta)\text{-gMod}$  and by  $R(\beta)\text{-gmod}$  the category of graded  $R(\beta)$ -modules and the category of graded  $R(\beta)$ -modules which are finite-dimensional over  $\mathbf{k}$ , respectively. We set

$$R\text{-gmod} = \bigoplus_{\beta \in \mathbf{Q}^+} R(\beta)\text{-gmod} \quad \text{and} \quad R\text{-mod} = \bigoplus_{\beta \in \mathbf{Q}^+} R(\beta)\text{-mod}.$$

For  $\beta, \gamma \in \mathbf{Q}^+$  with  $|\beta| = m$ ,  $|\gamma| = n$ , set

$$e(\beta, \gamma) = \sum_{\substack{\nu \in I^{\beta+\gamma}, \\ (\nu_1, \dots, \nu_m) \in I^\beta}} e(\nu) \in R(\beta + \gamma).$$

Then  $e(\beta, \gamma)$  is an idempotent. Let

$$(1.2) \quad R(\beta) \otimes R(\gamma) \rightarrow e(\beta, \gamma) R(\beta + \gamma) e(\beta, \gamma)$$

be the  $\mathbf{k}$ -algebra homomorphism given by  $e(\mu) \otimes e(\nu) \mapsto e(\mu * \nu)$  ( $\mu \in I^\beta$  and  $\nu \in I^\gamma$ )  $x_k \otimes 1 \mapsto x_k e(\beta, \gamma)$  ( $1 \leq k \leq m$ ),  $1 \otimes x_k \mapsto x_{m+k} e(\beta, \gamma)$  ( $1 \leq k \leq n$ ),  $\tau_k \otimes 1 \mapsto \tau_k e(\beta, \gamma)$  ( $1 \leq k < m$ ),  $1 \otimes \tau_k \mapsto \tau_{m+k} e(\beta, \gamma)$  ( $1 \leq k < n$ ). Here  $\mu * \nu$  is the concatenation of  $\mu$  and  $\nu$ ; i.e.,  $\mu * \nu = (\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n)$ .

For an  $R(\beta)$ -module  $M$  and an  $R(\gamma)$ -module  $N$ , we define the *convolution product*  $M \circ N$  by

$$M \circ N = R(\beta + \gamma) e(\beta, \gamma) \bigotimes_{R(\beta) \otimes R(\gamma)} (M \otimes N).$$

For  $M \in R(\beta)\text{-mod}$ , the dual space

$$M^* := \text{Hom}_{\mathbf{k}}(M, \mathbf{k})$$

admits an  $R(\beta)$ -module structure via

$$(r \cdot f)(u) := f(\psi(r)u) \quad (r \in R(\beta), u \in M),$$

where  $\psi$  denotes the  $\mathbf{k}$ -algebra anti-involution on  $R(\beta)$  which fixes the generators  $e(\nu)$ ,  $x_m$  and  $\tau_k$  for  $\nu \in I^\beta$ ,  $1 \leq m \leq |\beta|$  and  $1 \leq k < |\beta|$ .

It is known that (see [17, Theorem 2.2 (2)])

$$(1.3) \quad (M_1 \circ M_2)^* \simeq q^{(\beta, \gamma)} (M_2^* \circ M_1^*)$$

for any  $M_1 \in R(\beta)\text{-gmod}$  and  $M_2 \in R(\gamma)\text{-gmod}$ .

A simple module  $M$  in  $R\text{-gmod}$  is called *self-dual* if  $M^* \simeq M$ . Every simple module is isomorphic to a grading shift of a self-dual simple module ([13, §.3.2]).

Let us denote by  $K(R\text{-gmod})$  the Grothendieck group of  $R\text{-gmod}$ . Then,  $K(R\text{-gmod})$  is an algebra over  $\mathbf{A} := \mathbb{Z}[q^{\pm 1}]$  with the multiplication induced by the convolution product and the  $\mathbf{A}$ -action induced by the grading shift functor  $q$ .

In [13, 22], it is shown that KLR algebras *categorify* the negative half of the corresponding quantum group. More precisely, we have the following theorem.

**Theorem 1.3** ([13, 22]). *For a given Cartan datum  $(A, P, \Pi, P^\vee, \Pi^\vee)$ , we take a parameter matrix  $(Q_{ij})_{i,j \in J}$  satisfying the conditions in (1.1), and let  $U_q(\mathfrak{g})$  and  $R(\beta)$  be the associated quantum group and the KLR algebras, respectively. Then there exists an  $\mathbf{A}$ -algebra isomorphism*

$$(1.4) \quad U_{\mathbf{A}}^-(\mathfrak{g})^\vee \simeq K(R\text{-gmod}).$$

KLR algebras also categorify the upper global bases.

**Definition 1.4.** *We say that the KLR algebra  $R$  is symmetric if  $Q_{i,j}(u, v)$  is a polynomial in  $u - v$  for all  $i, j \in I$ .*

In particular, the generalized Cartan matrix  $A$  is symmetric.

**Theorem 1.5** ([24, 23]). *Assume that KLR algebra  $R$  is symmetric and the base field  $\mathbf{k}$  is of characteristic zero. Then under the isomorphism (1.4) in Theorem 1.3, the upper global basis corresponds to the set of the isomorphism classes of self-dual simple  $R$ -modules.*

### 1.3. R-matrices for Khovanov-Lauda-Rouquier algebras.

For  $|\beta| = n$  and  $1 \leq a < n$ , we define  $\varphi_a \in R(\beta)$  by

$$(1.5) \quad \varphi_a e(\nu) = \begin{cases} (\tau_a x_a - x_a \tau_a) e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \tau_a e(\nu) & \text{otherwise.} \end{cases}$$

They are called the *intertwiners*. Since  $\{\varphi_a\}_{1 \leq a < n}$  satisfies the braid relation,  $\varphi_w := \varphi_{i_1} \cdots \varphi_{i_\ell}$  does not depend on the choice of reduced expression  $w = s_{i_1} a \cdots s_{i_\ell}$ .

For  $m, n \in \mathbb{Z}_{\geq 0}$ , let us denote by  $w[m, n]$  the element of  $\mathfrak{S}_{m+n}$  defined by

$$(1.6) \quad w[m, n](k) = \begin{cases} k + n & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq m + n. \end{cases}$$

Let  $\beta, \gamma \in \mathbf{Q}^+$  with  $|\beta| = m$ ,  $|\gamma| = n$ , and let  $M$  be an  $R(\beta)$ -module and  $N$  an  $R(\gamma)$ -module. Then the map  $M \otimes N \rightarrow N \circ M$  given by  $u \otimes v \mapsto \varphi_{w[m, n]}(v \otimes u)$  is  $R(\beta) \otimes R(\gamma)$ -linear, and hence it extends to an  $R(\beta + \gamma)$ -module homomorphism

$$(1.7) \quad R_{M, N}: M \otimes N \longrightarrow N \circ M.$$

Assume that the KLR algebra  $R(\beta)$  is symmetric. Let  $z$  be an indeterminate which is homogeneous of degree 2, and let  $\psi_z$  be the graded algebra homomorphism

$$\psi_z: R(\beta) \rightarrow \mathbf{k}[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an  $R(\beta)$ -module  $M$ , we denote by  $M_z$  the  $(\mathbf{k}[z] \otimes R(\beta))$ -module  $\mathbf{k}[z] \otimes M$  with the action of  $R(\beta)$  twisted by  $\psi_z$ . Namely,

$$(1.8) \quad \begin{aligned} e(\nu)(a \otimes u) &= a \otimes e(\nu)u, \\ x_k(a \otimes u) &= (za) \otimes u + a \otimes (x_k u), \\ \tau_k(a \otimes u) &= a \otimes (\tau_k u) \end{aligned}$$

for  $\nu \in I^\beta$ ,  $a \in \mathbf{k}[z]$  and  $u \in M$ . For  $u \in M$ , we sometimes denote by  $u_z$  the corresponding element  $1 \otimes u$  of the  $R(\beta)$ -module  $M_z$ .

For a non-zero  $M \in R(\beta)\text{-mod}$  and a non-zero  $N \in R(\gamma)\text{-mod}$ ,

$$(1.9) \quad \begin{aligned} &\text{let } s \text{ be the order of zero of } R_{M_z, N}: M_z \circ N \longrightarrow N \circ M_z; \text{ i.e., the} \\ &\text{largest non-negative integer such that the image of } R_{M_z, N} \text{ is contained} \\ &\text{in } z^s(N \circ M_z). \end{aligned}$$

Note that such an  $s$  exists because  $R_{M_z, N}$  does not vanish ([11, Proposition 1.4.4 (iii)]). We denote by  $R_{M_z, N}^{\text{ren}}$  the morphism  $z^{-s}R_{M_z, N}$ .

**Definition 1.6.** Assume that  $R(\beta)$  is symmetric. For a non-zero  $M \in R(\beta)\text{-mod}$  and a non-zero  $N \in R(\gamma)\text{-mod}$ , let  $s$  be an integer as in (1.9). We define

$$\mathbf{r}_{M, N}: M \circ N \rightarrow N \circ M$$

by

$$\mathbf{r}_{M, N} = R_{M_z, N}^{\text{ren}}|_{z=0},$$

and call it the renormalized R-matrix.

By the definition, the renormalized R-matrix  $\mathbf{r}_{M, N}$  never vanishes.

We define also

$$\mathbf{r}_{N, M}: N \circ M \rightarrow M \circ N$$

by

$$\mathbf{r}_{N, M} = ((-z)^{-t}R_{N, M_z})|_{z=0},$$

where  $t$  is the order of zero of  $R_{N, M_z}$ .

If  $R(\beta)$  and  $R(\gamma)$  are symmetric, then  $s$  coincides with the multiplicity of zero of  $R_{M, N_z}$ , and  $(z^{-s}R_{M_z, N})|_{z=0} = ((-z)^{-s}R_{M, N_z})|_{z=0}$  (see, [12, (1.11)]).

By the construction, if the composition  $(N_1 \circ \mathbf{r}_{M, N_2}) \circ (\mathbf{r}_{M, N_1} \circ N_2)$  for  $M, N_1, N_2 \in R\text{-mod}$  doesn't vanish then it is equal to  $\mathbf{r}_{M, N_1 \circ N_2}$ .

**Definition 1.7.** A simple  $R(\beta)$ -module  $M$  is called real if  $M \circ M$  is simple.

**Theorem 1.8** ([12, Theorem 3.2]). Let  $\beta, \gamma \in \mathbf{Q}^+$  and assume that  $R(\beta)$  is symmetric. Let  $M$  be a real simple module in  $R(\beta)\text{-mod}$  and  $N$  a simple module in  $R(\gamma)\text{-mod}$ . Then

- (i)  $M \circ N$  and  $N \circ M$  have simple socles and simple heads.

- (ii) Moreover,  $\text{Im}(\mathbf{r}_{M,N})$  is equal to the head of  $M \circ N$  and socle of  $N \circ M$ . Similarly,  $\text{Im}(\mathbf{r}_{N,M})$  is equal to the head of  $N \circ M$  and socle of  $M \circ N$ .

## 2. SIMPLICITY OF HEADS AND SOCLES OF CONVOLUTION PRODUCTS

In the rest of this paper, we assume that  $R(\beta)$  is symmetric for any  $\beta \in \mathbf{Q}^+$ , i.e.,  $Q_{ij}(u, v)$  is a function in  $u - v$  for any  $i, j \in I$ .

We also work always in the category of graded modules. For the sake of simplicity, we simply say that  $M$  is an  $R$ -module instead of saying that  $M$  is a graded  $R(\beta)$ -module for  $\beta \in \mathbf{Q}^+$ . We also sometimes ignore grading shifts if there is no afraid of confusion. Hence, for  $R$ -modules  $M$  and  $N$ , we sometimes say that  $f: M \rightarrow N$  is a homomorphism if  $f: q^a M \rightarrow N$  is a morphism in  $R\text{-gmod}$  for some  $a \in \mathbb{Z}$ . If we want to emphasize that  $f: q^a M \rightarrow N$  is a morphism in  $R\text{-gmod}$ , we say so.

**Proposition 2.1.** *Let  $M$  be a real simple module, and let  $N$  be a module with a simple socle. If the following diagram*

$$\begin{array}{ccc} \text{soc}(N) \circ M & \xrightarrow{\mathbf{r}_{\text{soc}(N), M}} & M \circ \text{soc}(N) \\ \downarrow & & \downarrow \\ N \circ M & \xrightarrow{\mathbf{r}_{N, M}} & M \circ N \end{array}$$

*commutes up to a constant multiple, then  $\text{soc}(M \circ \text{soc}(N))$  is equal to the socle of  $M \circ N$ . In particular  $M \circ N$  has a simple socle.*

*Proof.* Let  $S$  be an arbitrary simple submodule of  $M \circ N$ . Then we have a commutative diagram

$$\begin{array}{ccc} S \circ M & \xrightarrow{\quad} & M \circ S \\ \downarrow & & \downarrow \\ M \circ N \circ M & \xrightarrow{M \circ \mathbf{r}_{N, M}} & M \circ M \circ N. \end{array}$$

Hence  $S \circ M \subset M \circ (\mathbf{r}_{N, M})^{-1}(S)$ . Hence there exists a submodule  $K$  of  $N$  such that  $S \subset M \circ K$  and  $K \circ M \subset (\mathbf{r}_{N, M})^{-1}(S)$ . Hence  $K \neq 0$  and  $\text{soc}(N) \subset K$ . Hence  $\mathbf{r}_{N, M}(\text{soc}(N) \circ M) \subset \mathbf{r}_{N, M}(K \circ M) \subset S$ . Since  $\mathbf{r}_{N, M}(\text{soc}(N) \circ M)$  is non-zero by the assumption, we have  $\mathbf{r}_{N, M}(\text{soc}(N) \circ M) = S$ . Hence we obtain the desired result.  $\square$

The following is a dual form of the preceding proposition.

**Proposition 2.2.** *Let  $M$  be a real simple module. Let  $N$  be a module with a simple head. If the following diagram*

$$\begin{array}{ccc} M \circ N & \xrightarrow{\mathbf{r}_{M,N}} & N \circ M \\ \downarrow & & \downarrow \\ M \circ \text{hd}(N) & \xrightarrow{\mathbf{r}_{M,\text{hd}(N)}} & \text{hd}(N) \circ M \end{array}$$

*commutes up to a constant multiple, then  $M \diamond \text{hd}(N)$  is equal to the simple head of  $M \circ N$ .*

**Definition 2.3.** *For non-zero  $M, N \in R\text{-gmod}$ , we denote by  $\Lambda(M, N)$  the homogeneous degree of the  $R$ -matrix  $\mathbf{r}_{M,N}$ .*

Hence

$$\begin{aligned} R_{M_z,N}^{\text{ren}} : M_z \circ N &\rightarrow q^{-\Lambda(M,N)} N \circ M_z \quad \text{and} \\ \mathbf{r}_{M,N} : M \circ N &\rightarrow q^{-\Lambda(M,N)} N \circ M \end{aligned}$$

are morphisms in  $R\text{-gMod}$  and in  $R\text{-gmod}$ , respectively.

**Definition 2.4.** *For simple  $R$ -modules  $M$  and  $N$ , we denote by  $M \diamond N$  the head of  $M \circ N$ .*

**Lemma 2.5.** *For non-zero  $R$ -modules  $M$  and  $N$ , we have*

$$\Lambda(M, N) \equiv (\text{wt}(M), \text{wt}(N)) \pmod{2}.$$

*Proof.* Set  $\beta := -\text{wt}(M)$  and  $\gamma := -\text{wt}(N)$ . By [11, (1.3.3)], the homogeneous degree of  $R_{M_z,N}$  is  $-(\beta, \gamma) + 2(\beta, \gamma)_n$ , where  $(\bullet, \bullet)_n$  is the symmetric bilinear form on  $\mathbb{Q}$  given by  $(\alpha_i, \alpha_j)_n = \delta_{ij}$ . Hence  $R_{M_z,N}^{\text{ren}} = z^{-s} R_{M_z,N}$  has degree  $-(\beta, \gamma) + 2(\beta, \gamma)_n - 2s$ .  $\square$

**Definition 2.6.** *For non-zero  $R$ -modules  $M$  and  $N$ , we set*

$$\tilde{\Lambda}(M, N) := \frac{1}{2}(\Lambda(M, N) + (\text{wt}(M), \text{wt}(N))).$$

**Lemma 2.7.** *Let  $M$  and  $N$  be self-dual simple modules. If one of them is real, then*

$$q^{\tilde{\Lambda}(M,N)} M \diamond N$$

*is a self-dual simple module.*

*Proof.* Set  $\beta = \text{wt}(M)$  and  $\gamma = \text{wt}(N)$ . Set  $M \diamond N = q^c L$  for some self-dual simple module  $L$  and some  $c \in \mathbb{Z}$ . Then we have

$$M \circ N \twoheadrightarrow q^c L \hookrightarrow q^{-\Lambda(M,N)} N \circ M.$$

Taking dual, we obtain

$$q^{\Lambda(M,N)+(\beta,\gamma)} M \circ N \twoheadrightarrow q^{-c} L \hookrightarrow q^{(\beta,\gamma)} N \circ M.$$

Hence we have  $c = -c - \Lambda(M, N) - (\beta, \gamma)$ , which implies  $c = -\tilde{\Lambda}(M, N)$ .  $\square$

**Lemma 2.8.** *Let  $M$  be a simple module and let  $N_1, N_2$  be non-zero modules. Then the composition*

$$M \circ N_1 \circ N_2 \xrightarrow{\mathbf{r}_{M, N_1} \circ N_2} N_1 \circ M \circ N_2 \xrightarrow{N_1 \circ \mathbf{r}_{M, N_2}} N_1 \circ N_2 \circ M$$

*coincides with  $\mathbf{r}_{M, N_1 \circ N_2}$ , and the composition*

$$N_1 \circ N_2 \circ M \xrightarrow{N_1 \circ \mathbf{r}_{N_2, M}} N_1 \circ M \circ N_2 \xrightarrow{\mathbf{r}_{N_1, M} \circ N_2} M \circ N_1 \circ N_2$$

*coincides with  $\mathbf{r}_{N_1 \circ N_2, N}$ .*

*In particular, we have*

$$\Lambda(M, N_1 \circ N_2) = \Lambda(M, N_1) + \Lambda(M, N_2)$$

*and*

$$\Lambda(N_1 \circ N_2, M) = \Lambda(N_1, M) + \Lambda(N_2, M).$$

*Proof.* It is enough to show that the compositions  $(N_1 \circ \mathbf{r}_{M, N_2}) \circ (\mathbf{r}_{M, N_1} \circ N_2)$  and  $(\mathbf{r}_{N_1, M} \circ N_2) \circ (N_1 \circ \mathbf{r}_{N_2, M})$  do not vanish.

Assume that  $(N_1 \circ \mathbf{r}_{M, N_2}) \circ (\mathbf{r}_{M, N_1} \circ N_2)$  vanishes. Then we have

$$\text{Im}(\mathbf{r}_{M, N_1}) \circ N_2 \subset N_1 \circ \text{Ker}(\mathbf{r}_{M, N_2}).$$

By [12, Lemma 3.1], there is a submodule  $Z$  of  $M$  such that

$$\text{Im}(\mathbf{r}_{M, N_1}) \subset N_1 \circ Z \quad \text{and} \quad Z \circ N_2 \subset \text{Ker}(\mathbf{r}_{M, N_2}).$$

It contradicts the assumption that  $M$  is simple.

Similarly, one can show that  $(\mathbf{r}_{N_1, M} \circ N_2) \circ (N_1 \circ \mathbf{r}_{N_2, M})$  does not vanish.  $\square$

**Lemma 2.9.** *Let  $M$  and  $N$  be simple  $R$ -modules. Then we have*

- (i)  $\Lambda(M, N) + \Lambda(N, M) \in 2\mathbb{Z}_{\geq 0}$ .
- (ii) *If  $\Lambda(M, N) + \Lambda(N, M) = 2m$  for some  $m \in \mathbb{Z}_{\geq 0}$ , then*

$$R_{M_z, N}^{\text{ren}} \circ R_{N, M_z}^{\text{ren}} = z^m \text{id}_{N \circ M_z} \quad \text{and} \quad R_{N, M_z}^{\text{ren}} \circ R_{M_z, N}^{\text{ren}} = z^m \text{id}_{M_z \circ N}$$

*up to constant multiples.*

*Proof.* By [11, Proposition 1.6.2], the morphism

$$R_{N, M_z}^{\text{ren}} \circ R_{M_z, N}^{\text{ren}} : M_z \circ N \rightarrow M_z \circ N$$

is equal to  $f(z) \text{id}_{M_z \circ N}$  for some  $0 \neq f(z) \in \mathbf{k}[z]$ . Since  $R_{N, M_z}^{\text{ren}} \circ R_{M_z, N}^{\text{ren}}$  is homogeneous of degree  $\Lambda(M, N) + \Lambda(N, M)$ , we have  $f(z) = cz^{\frac{1}{2}(\Lambda(M, N) + \Lambda(N, M))}$  for some  $c \in \mathbf{k}^\times$ .  $\square$



**Definition 2.10.** For non-zero modules  $M$  and  $N$ , we set

$$\mathfrak{d}(M, N) = \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M)).$$

Note that if  $M$  and  $N$  are simple modules, then we have  $\mathfrak{d}(M, N) \in \mathbb{Z}_{\geq 0}$ . Note also that if  $M, N_1, N_2$  are simple modules, then we have  $\mathfrak{d}(M, N_1 \circ N_2) = \mathfrak{d}(M, N_1) + \mathfrak{d}(M, N_2)$ .

**Lemma 2.11** ([12]). *Let  $M, N$  be simple modules and assume that one of them is real. Then the following conditions are equivalent.*

- (a)  $\mathfrak{d}(M, N) = 0$ .
- (b)  $\mathbf{r}_{M,N}$  and  $\mathbf{r}_{N,M}$  are inverse to each other up to a constant multiple.
- (c)  $M \circ N$  and  $N \circ M$  are isomorphic up to a grading shift.
- (d)  $M \diamond N$  and  $N \diamond M$  are isomorphic up to a grading shift.
- (e)  $M \circ N$  is simple.

**Definition 2.12.** Let  $M, N$  be simple modules.

- (i) We say that  $M$  and  $N$  commute if  $\mathfrak{d}(M, N) = 0$ .
- (ii) We say that  $M$  and  $N$  simply-linked if  $\mathfrak{d}(M, N) = 1$ .

**Proposition 2.13.** Let  $M_1, \dots, M_r$  be a commuting family of real simple modules. Then the convolution product

$$M_1 \circ \dots \circ M_r$$

is a real simple module.

*Proof.* We shall first show the simplicity of the convolutions. By induction on  $r$ , we may assume that  $M_2 \circ \dots \circ M_r$  is a simple module. Then we have

$$\mathfrak{d}(M_1, M_2 \circ \dots \circ M_r) = \sum_{s=2}^r \mathfrak{d}(M_1, M_s) = 0$$

so that  $M_1 \circ \dots \circ M_r$  is simple.

Since  $(M_1 \circ \dots \circ M_r) \circ (M_1 \circ \dots \circ M_r)$  is also simple,  $M_1 \circ \dots \circ M_r$  is real.  $\square$

**Definition 2.14.** Let  $M_1, \dots, M_m$  be real simple modules. Assume that they are commuting with each other. We set

$$\bigodot_{1 \leq k \leq m} M_k := q^{\sum_{1 \leq i < j \leq m} \tilde{\Lambda}(M_i, M_j)} M_1 \circ \dots \circ M_m.$$

**Lemma 2.15.** Let  $M_1, \dots, M_m$  be real simple modules commuting with each other. Then for any  $\sigma \in \mathfrak{S}_m$ , we have

$$\bigodot_{1 \leq k \leq m} M_k \simeq \bigodot_{1 \leq k \leq m} M_{\sigma(k)} \quad \text{in } R\text{-gmod}.$$

Moreover, if the  $M_k$ 's are self-dual, then so is  $\bigodot_{1 \leq k \leq m} M_k$ .

*Proof.* It follows from Lemma 2.7 and  $q^{\tilde{\Lambda}(M_i, M_j)} M_i \circ M_j \simeq q^{\tilde{\Lambda}(M_j, M_i)} M_j \circ M_i$ .  $\square$

**Proposition 2.16.** *Let  $f : N_1 \rightarrow N_2$  be a morphism between non-zero modules  $R$ -modules  $N_1, N_2$  and let  $M$  be a non-zero  $R$ -module.*

(i) *If  $\Lambda(M, N_1) = \Lambda(M, N_2)$ , then the following diagram is commutative:*

$$\begin{array}{ccc} M \circ N_1 & \xrightarrow{r_{M, N_1}} & N_1 \circ M \\ M \circ f \downarrow & & \downarrow f \circ M \\ M \circ N_2 & \xrightarrow{r_{M, N_2}} & N_2 \circ M. \end{array}$$

(ii) *If  $\Lambda(M, N_1) < \Lambda(M, N_2)$ , then the composition*

$$M \circ N_1 \xrightarrow{M \circ f} M \circ N_2 \xrightarrow{r_{M, N_2}} N_2 \circ M$$

*vanishes.*

(iii) *If  $\Lambda(M, N_1) > \Lambda(M, N_2)$ , then the composition*

$$M \circ N_1 \xrightarrow{r_{M, N_1}} N_1 \circ M \xrightarrow{f \circ M} N_2 \circ M$$

*vanishes.*

(iv) *If  $f$  is surjective, then we have*

$$\Lambda(M, N_1) \geq \Lambda(M, N_2) \quad \text{and} \quad \Lambda(N_1, M) \geq \Lambda(N_2, M)$$

*If  $f$  is injective, then we have*

$$\Lambda(M, N_1) \leq \Lambda(M, N_2) \quad \text{and} \quad \Lambda(N_1, M) \leq \Lambda(N_2, M)$$

*Proof.* Let  $s_i$  be the order of zero of  $R_{M_z, N_i}$  for  $i = 1, 2$ . Then we have  $\Lambda(M, N_1) - \Lambda(M, N_2) = 2(s_2 - s_1)$ .

Set  $m := \min\{s_1, s_2\}$ . Then the following diagram is commutative:

$$\begin{array}{ccc} M_z \circ N_1 & \xrightarrow{z^{-m} R_{M_z, N_1}} & N_1 \circ M_z \\ M_z \circ f \downarrow & & \downarrow f \circ M_z \\ M_z \circ N_2 & \xrightarrow{z^{-m} R_{M_z, N_2}} & N_2 \circ M_z. \end{array}$$

(i) If  $s_1 = s_2$ , then by specializing  $z = 0$  in the above diagram, we obtain the commutativity of the diagram in (i).

(ii) If  $s_1 > s_2$ , then we have

$$z^{-m} R_{M_z, N_1} = z^{s_1 - m} (z^{-s_1} R_{M_z, N_1})$$

so that  $z^{-m} R_{M_z, N_1}|_{z=0}$  vanishes. Hence we have

$$r_{M, N_2} \circ (M \circ f) = z^{-m} R_{M_z, N_2}|_{z=0} \circ (M \circ f) = 0,$$

as desired. In particular,  $f$  is not surjective.

(iii) Similarly, if  $s_1 < s_2$ , then we have  $(f \circ M) \circ \mathbf{r}_{M, N_1} = 0$ , and  $f$  is not injective.

(iv) The statements for  $\Lambda(M, N_1)$  and  $\Lambda(N_1, M)$  follow from (ii) and (iii). The other statements can be shown in a similar way.  $\square$

**Proposition 2.17.** *Let  $L$ ,  $M$  and  $N$  be simple modules. Then we have*

$$(2.1) \quad \begin{aligned} \Lambda(L, S) &\leq \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(S, L) \leq \Lambda(M, L) + \Lambda(N, L) \text{ and} \\ \mathfrak{d}(S, L) &\leq \mathfrak{d}(M, L) + \mathfrak{d}(N, L) \end{aligned}$$

for any subquotient  $S$  of  $M \circ N$ . Moreover, when  $L$  is real, the following conditions are equivalent.

- (a)  $L$  commutes with  $M$  and  $N$ .
- (b) Any simple subquotient  $S$  of  $M \circ N$  commutes with  $L$  and satisfies  $\Lambda(L, S) = \Lambda(L, M) + \Lambda(L, N)$ .
- (c) Any simple subquotient  $S$  of  $M \circ N$  commutes with  $L$  and satisfies  $\Lambda(S, L) = \Lambda(M, L) + \Lambda(N, L)$ .

*Proof.* The inequalities (2.1) are consequences of the preceding proposition. Let us show the equivalence of (a)–(c).

Let  $M \circ N = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0$  be a Jordan-Hölder series of  $M \circ N$ . Then the renormalized R-matrix  $R_{L_z, M \circ N}^{\text{ren}} = (M \circ R_{L_z, N}^{\text{ren}}) \circ (R_{L_z, M}^{\text{ren}} \circ N): L_z \circ M \circ N \rightarrow M \circ N \circ L_z$  is homogeneous of degree  $\Lambda(L, M) + \Lambda(L, N)$  and it sends  $L_z \circ K_k$  to  $K_k \circ L_z$  for any  $k \in \mathbb{Z}$ . Hence  $f := \mathbf{r}_{L, M \circ N} = R_{L_z, M \circ N}^{\text{ren}}|_{z=0}$  sends  $L \circ K_k$  to  $K_k \circ L$ .

First assume (a). Then  $f$  is an isomorphism. Hence  $f|_{L \circ K_k}: L \circ K_k \rightarrow K_k \circ L$  is injective. By comparing their dimension,  $f|_{L \circ K_k}$  is an isomorphism. Hence  $f|_{L \circ (K_k/K_{k+1})}$  is an isomorphism of homogeneous degree  $\Lambda(L, M) + \Lambda(L, N)$ . Hence we obtain (b).

Conversely, assume (b). Then,  $R_{L_z, M \circ N}^{\text{ren}}|_{L_z \circ (K_k/K_{k+1})}$  and  $R_{L_z, K_k/K_{k+1}}^{\text{ren}}$  have the same homogeneous degree, and hence they should coincide. It implies that  $f|_{L \circ (K_k/K_{k+1})} = \mathbf{r}_{L, K_k/K_{k+1}}$  is an isomorphism for any  $k$ . Therefore  $f = (M \circ \mathbf{r}_{L, N}) \circ (\mathbf{r}_{L, M} \circ N)$  is an isomorphism, which implies that  $\mathbf{r}_{L, N}$  and  $\mathbf{r}_{L, M}$  are isomorphisms. Thus we obtain (a).

Similarly (a) and (c) are equivalent.  $\square$

**Lemma 2.18.** *Let  $L$ ,  $M$  and  $N$  be simple modules. We assume that  $L$  is real and commutes with  $M$ . Then the diagram*

$$\begin{array}{ccc} L \circ (M \circ N) & \xrightarrow{\mathbf{r}_{L, M \circ N}} & (M \circ N) \circ L \\ \downarrow & & \downarrow \\ L \circ (M \diamond N) & \xrightarrow{\mathbf{r}_{L, M \diamond N}} & (M \diamond N) \circ L \end{array}$$

commutes.

*Proof.* Otherwise the composition

$$L \circ M \circ N \xrightarrow[\mathbf{r}_{L,M \circ N}]{\sim} M \circ L \circ N \xrightarrow{M \circ \mathbf{r}_{L,N}} M \circ N \circ L \longrightarrow (M \diamond N) \circ L$$

vanishes. Hence we have

$$M \circ \text{Im}(\mathbf{r}_{L,N}) \subset \text{Ker}(M \circ N \rightarrow M \diamond N) \circ L.$$

Hence there exists a submodule  $K$  of  $N$  such that

$$\text{Im}(\mathbf{r}_{L,N}) \subset K \circ L \text{ and } M \circ K \subset \text{Ker}(M \circ N \rightarrow M \diamond N).$$

The first inclusion implies  $K \neq 0$  and the second implies  $K \neq N$ , which contradicts the simplicity of  $N$ .  $\square$

The following lemma can be proved similarly.

**Lemma 2.19.** *Let  $L$ ,  $M$  and  $N$  be simple modules. We assume that  $L$  is real and commutes with  $N$ . Then the diagram*

$$\begin{array}{ccc} (M \circ N) \circ L & \xrightarrow{\mathbf{r}_{M \circ N, L}} & L \circ (M \circ N) \\ \downarrow & & \downarrow \\ (M \diamond N) \circ L & \xrightarrow{\mathbf{r}_{M \diamond N, L}} & L \circ (M \diamond N) \end{array}$$

*commutes.*

The following proposition follows from Lemma 2.18 and Lemma 2.19.

**Proposition 2.20.** *Let  $L$ ,  $M$  and  $N$  be simple modules. Assume that  $L$  is real. Then we have*

(i) *If  $L$  and  $M$  commute, then*

$$\Lambda(L, M \diamond N) = \Lambda(L, M) + \Lambda(L, N).$$

(ii) *If  $L$  and  $N$  commute, then*

$$\Lambda(M \diamond N, L) = \Lambda(M, L) + \Lambda(N, L).$$

**Proposition 2.21.** *Let  $X, Y, M$  and  $N$  be simple  $R$ -modules. Assume that there is an exact sequence*

$$0 \rightarrow X \rightarrow M \circ N \rightarrow Y \rightarrow 0,$$

*$X \circ N$  and  $Y \circ N$  are simple and  $X \circ N \not\cong Y \circ N$  as ungraded modules. Then  $N$  is a real simple module.*

*Proof.* Assume that  $N$  is not real. Then  $N \circ N$  is reducible and we have  $\mathbf{r}_{N,N} \neq c \text{id}_{N \circ N}$  for any  $c \in \mathbf{k}$ . Note that  $N \circ N$  is of length 2, because  $M \circ N \circ N$  is of length 2.

Let  $S$  be a simple submodule of  $N \circ N$ . Consider an exact sequence

$$0 \longrightarrow X \circ N \longrightarrow M \circ N \circ N \longrightarrow Y \circ N \longrightarrow 0.$$

Then we have

$$(2.2) \quad (X \circ N) \cap (M \circ S) = 0.$$

Indeed, if  $(X \circ N) \subset (M \circ S)$ , then there exists a submodule  $Z$  of  $N$  such that  $X \subset M \circ Z$  and  $Z \circ N \subset S$ . It contradicts the simplicity of  $N$ . Thus (2.2) holds.

Note that (2.2) implies that

$$M \circ S \simeq Y \circ N.$$

(a) Assume first that  $N \circ N$  is semisimple so that  $N \circ N = S \oplus S'$  for some simple submodule  $S'$  of  $N \circ N$ . Then  $M \circ S \simeq Y \circ N \simeq M \circ S'$ . Hence  $M \circ S \simeq X \circ N \simeq M \circ S'$ . Therefore we obtain  $X \circ N \simeq Y \circ N$ , which is a contradiction.

(b) Assume that  $N \circ N$  is not semisimple so that  $S$  is a unique non-zero proper submodule of  $N \circ N$  and  $(N \circ N)/S$  is a unique non-zero proper quotient of  $N \circ N$ . Without loss of generality, we may assume that  $\mathbf{k}$  is algebraically closed. Let  $x \in \mathbf{k}$  be an eigenvalue of  $\mathbf{r}_{N,N}$ . Since  $\mathbf{r}_{N,N} \notin \mathbf{k} \text{id}_{N \circ N}$ , we have  $0 \subsetneq \text{Im}(\mathbf{r}_{N,N} - x \text{id}_{N \circ N}) \subsetneq N \circ N$ . It follows that

$$S = \text{Im}(\mathbf{r}_{N,N} - x \text{id}_{N \circ N}) \simeq (N \circ N)/S,$$

and hence we have an exact sequence

$$0 \longrightarrow M \circ S \longrightarrow M \circ N \circ N \longrightarrow M \circ ((N \circ N)/S) \longrightarrow 0.$$

Since  $M \circ N \circ N$  is of length 2, we have

$$X \circ N \simeq M \circ S \simeq M \circ ((N \circ N)/S) \simeq Y \circ N,$$

which is a contradiction.  $\square$

**Corollary 2.22.** *Let  $X, Y, N$  be simple  $R$ -modules and let  $M$  be a real simple  $R$ -module. If we have an exact sequence*

$$0 \rightarrow X \rightarrow M \circ N \rightarrow Y \rightarrow 0$$

*and if  $X \circ N$  and  $Y \circ N$  are simple, then  $N$  is a real simple module.*

*Proof.* Since  $M$  is real and  $M \circ N$  is not simple,  $X$  is not isomorphic to  $Y$  as an ungraded module. It follows that  $X \circ N$  is not isomorphic to  $Y \circ N$ , because  $K(R\text{-gmod})$  is a domain so that  $[X \circ N] = q^m[Y \circ N]$  for some  $m \in \mathbb{Z}$  implies  $[X] = q^m[Y]$ . Now the assertion follows from the above proposition.  $\square$

The following lemmas are used later.

**Lemma 2.23.** *Let  $M$  and  $N$  be real simple modules. Assume that  $M \diamond N$  is real and commutes with  $N$ . Then for any  $n \in \mathbb{Z}_{\geq 0}$ , we have*

$$M^{\circ n} \diamond N^{\circ n} \simeq (M \diamond N)^{\circ n} \quad \text{as ungraded modules.}$$

*Proof.* Set  $L = M \diamond N$ . Since  $L^{\circ n}$  is simple, it is enough to give an epimorphism  $M^{\circ n} \diamond N^{\circ n} \twoheadrightarrow L^{\circ n}$ . We shall show it by induction on  $n$ . For  $n > 0$ , we have

$$\begin{aligned} M^{\circ n} \diamond N^{\circ n} &\simeq M^{\circ(n-1)} \diamond M \diamond N \diamond N^{\circ(n-1)} \\ &\twoheadrightarrow M^{\circ(n-1)} \diamond L \diamond N^{\circ(n-1)} \simeq M^{\circ(n-1)} \diamond N^{\circ(n-1)} \diamond L \twoheadrightarrow L^{\circ(n-1)} \diamond L. \end{aligned}$$

□

**Corollary 2.24.** *Let  $M$  and  $N$  be simple module. Assume that one of them is real. If there is an exact sequence*

$$0 \rightarrow q^m X \rightarrow M \circ N \rightarrow q^n Y \rightarrow 0$$

*for self-dual simple modules  $X, Y$  and integers  $m, n$ , then we have*

$$\mathfrak{d}(M, N) = m - n \geq 0.$$

*Proof.* We may assume that  $M$  and  $N$  are self-dual without loss of generality. Then we have  $n = -\tilde{\Lambda}(M, N)$ . Since  $q^m X \simeq q^{\Lambda(N, M)} N \diamond M \simeq q^{\Lambda(N, M) - \tilde{\Lambda}(N, M)} (q^{\tilde{\Lambda}(N, M)} N \diamond M)$ , we have  $m = \Lambda(N, M) - \tilde{\Lambda}(N, M)$ . Thus we obtain

$$m - n = \Lambda(N, M) - \tilde{\Lambda}(N, M) + \tilde{\Lambda}(M, N) = \mathfrak{d}(M, N).$$

□

### 3. LECLERC CONJECTURE

Recall that  $R$  is assumed to be a symmetric KLR algebra over a base field  $\mathbf{k}$ .

#### 3.1. Leclerc conjecture.

**Theorem 3.1.** *Let  $M$  and  $N$  be simple modules. We assume that  $M$  is real. Then we have the equalities in the Grothendieck group  $K(R\text{-gmod})$ :*

- (i)  $[M \circ N] = [M \diamond N] + \sum_k [S_k]$   
with simple modules  $S_k$  such that  $\Lambda(M, S_k) < \Lambda(M, M \diamond N) = \Lambda(M, N)$ ,
- (ii)  $[M \circ N] = [q^{\Lambda(N, M)} N \diamond M] + \sum_k [S_k]$   
with simple modules  $S_k$  such that  $\Lambda(S_k, M) < \Lambda(N \diamond M, M) = \Lambda(N, M)$ ,
- (iii)  $[N \circ M] = [N \diamond M] + \sum_k [S_k]$   
with simple modules  $S_k$  such that  $\Lambda(S_k, M) < \Lambda(N \diamond M, M) = \Lambda(N, M)$ , and
- (iv)  $[N \circ M] = [q^{\Lambda(M, N)} M \diamond N] + \sum_k [S_k]$   
with simple modules  $S_k$  such that  $\Lambda(M, S_k) < \Lambda(M, M \diamond N) = \Lambda(M, N)$ .

In particular,  $M \diamond N$  as well as  $N \diamond M$  appears only once in the Jordan-Hölder series of  $M \circ N$  in  $R\text{-mod}$ .

The following result is an immediate consequence of this theorem.

**Corollary 3.2.** *Let  $M$  and  $N$  be simple modules. We assume that one of them is real. Assume that  $M$  and  $N$  do not commute, Then we have the equality in the Grothendieck group  $K(R\text{-gmod})$*

$$[M \circ N] = [M \diamond N] + [q^{\Lambda(N,M)} N \diamond M] + \sum_k [S_k]$$

with simple modules  $S_k$ . Moreover we have

- (i) If  $M$  is real, then we have  $\Lambda(M, N \diamond M) < \Lambda(M, N)$ ,  $\Lambda(M \diamond N, M) < \Lambda(N, M)$  and  $\Lambda(M, S_k) < \Lambda(M, N)$ ,  $\Lambda(S_k, M) < \Lambda(N, M)$ .
- (ii) If  $N$  is real, then we have  $\Lambda(N, M \diamond N) < \Lambda(N, M)$ ,  $\Lambda(N, M \diamond N) < \Lambda(N, M)$  and  $\Lambda(N, S_k) < \Lambda(N, M)$ ,  $\Lambda(S_k, N) < \Lambda(M, N)$ .

*Proof of Theorem 3.1.* We shall prove only (i). The other statements are proved similarly.

Let us take a Jordan-Hölder series of  $M \circ N$

$$M \circ N = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0.$$

Then we have  $K_0/K_1 \simeq M \diamond N$ . Let us consider the renormalized R-matrix  $R_{M_z, M \circ N}^{\text{ren}} = (M \circ R_{M_z, N}^{\text{ren}}) \circ (R_{M_z, M}^{\text{ren}} \circ N)$

$$M_z \circ M \circ N \xrightarrow{R_{M_z, M \circ N}^{\text{ren}}} M \circ M_z \circ N \xrightarrow{M \circ R_{M_z, N}^{\text{ren}}} M \circ N \circ M_z.$$

Then  $R_{M_z, M \circ N}^{\text{ren}}$  sends  $M_z \circ K_k$  to  $K_k \circ M_z$  for any  $k$ . Hence evaluating the above diagram at  $z = 0$ , we obtain

$$\begin{array}{ccc} M \circ M \circ N & \xrightarrow{M \circ \mathbf{r}_{M, N}} & M \circ N \circ M \\ \uparrow & & \uparrow \\ M \circ K_1 & \longrightarrow & K_1 \circ M. \end{array}$$

Since  $\text{Im}(\mathbf{r}_{M, N}: M \circ N \rightarrow N \circ M) \simeq (M \circ N)/K_1$ , we have  $\mathbf{r}_{M, N}(K_1) = 0$ . Hence,  $R_{M_z, M \circ N}^{\text{ren}}$  sends  $M_z \circ K_1$  to  $(K_1 \circ M_z) \cap z((M \circ N) \circ M_z) = z(K_1 \circ M_z)$ . Thus  $z^{-1}R_{M_z, M \circ N}^{\text{ren}}|_{M_z \circ K_1}$  is well defined. Then it sends  $M_z \circ K_k$  to  $K_k \circ M_z$  for  $k \geq 1$ . Thus we obtain an R-matrix

$$z^{-1}R_{M_z, M \circ N}^{\text{ren}}|_{M_z \circ (K_k/K_{k+1})}: M_z \circ (K_k/K_{k+1}) \rightarrow (K_k/K_{k+1}) \circ M_z \quad \text{for } 1 \leq k \leq \ell.$$



Hence we have

$$R_{M_z, K_k/K_{k+1}}^{\text{ren}} = z^{-s_k} z^{-1} R_{M_z, M \circ N}^{\text{ren}}|_{M_z \circ (K_k/K_{k+1})}$$

for some  $s_k \in \mathbb{Z}_{\geq 0}$ . Since the homogeneous degree of  $R_{M_z, M \circ N}^{\text{ren}}$  is  $\Lambda(M, M \circ N) = \Lambda(M, N)$ , we obtain

$$\Lambda(M, K_k/K_{k+1}) = \Lambda(M, N) - 2(1 + s_k) < \Lambda(M, N).$$

□

The following theorem is an application of the above theorem.

**Theorem 3.3.** *Let  $\phi$  be an element of the Grothendieck group  $K(R\text{-gmod})$  given by*

$$\phi = \sum_{b \in B(\infty)} a_b [L_b],$$

where  $L_b$  is the self-dual simple module corresponding to  $b$  and  $a_b \in \mathbb{Z}[q^{\pm 1}]$ . Let  $A$  be a real simple module in  $R\text{-gmod}$ . Assume that we have an equality

$$\phi[A] = q^l [A] \phi$$

in  $K(R\text{-gmod})$  for some  $l \in \mathbb{Z}$ . Then  $A$  commutes with  $L_b$  and

$$l = \Lambda(A, L_b)$$

for every  $b \in B(\infty)$  such that  $a_b \neq 0$ .

*Proof.* Note that we have

$$\begin{aligned} \phi[A] &= \sum_b a_b [L_b \circ A] = \sum_b a_b ([L_b \diamond A] + \sum_k [S_{b,k}]) \quad \text{and} \\ q^l [A] \phi &= q^l \sum_b a_b [A \circ L_b] = q^l \sum_b a_b (q^{\Lambda(L_b, A)} [L_b \diamond A] + \sum_k [S^{b,k}]), \end{aligned}$$

for some simple modules  $S_{b,k}$  and  $S^{b,k}$  satisfying

$$\Lambda(S_{b,k}, A) < \Lambda(L_b, A) \quad \text{and} \quad \Lambda(S^{b,k}, A) < \Lambda(L_b, A).$$

We may assume that  $\{b \in B(\infty) \mid a_b \neq 0\} \neq \emptyset$ . Set

$$t := \max \{ \Lambda(L_b, A) \mid a_b \neq 0 \}.$$

By taking the classes of self-dual simple modules  $S$  with  $\Lambda(S, A) = t$  in the expansions of  $\phi[A]$  and  $q^l [A] \phi$ , we obtain

$$\sum_{\Lambda(L_b, A)=t} a_b [L_b \diamond A] = \sum_{\Lambda(L_b, A)=t} q^l a_b q^{\Lambda(L_b, A)} [L_b \diamond A].$$

In particular, we have  $t = -l$ .

Set

$$t' := \max \{ \Lambda(A, L_b) \mid a_b \neq 0 \}.$$

Then, by a similar argument we have  $t' = l$ .  
It follows that

$$0 = t + t' \geq \Lambda(L_b, A) + \Lambda(A, L_b) \geq 0$$

for every  $b$  such that  $a_b \neq 0$ . Hence  $A$  and  $L_b$  commute.

Since

$$\sum a_b q^{\Lambda(A, L_b)} [A \circ L_b] = \sum a_b [L_b \circ A] = \phi[A] = q^l [A] \phi = q^l \sum a_b [A \circ L_b],$$

we have

$$l = \Lambda(A, L_b)$$

for any  $b$  such that  $a_b \neq 0$ , as desired.  $\square$

**Corollary 3.4.** *Let  $M$  and  $N$  be simple modules. Assume that one of them is real. If  $[M]$  and  $[N]$   $q$ -commute (i.e.,  $[M][N] = q^n [N][M]$  for some  $n \in \mathbb{Z}$ ), then  $M$  and  $N$  commute. In particular,  $M \circ N$  is simple.*

The following corollary is an immediate consequence of the corollary above and Theorem 1.5.

**Corollary 3.5.** *Assume that the generalized Cartan matrix  $A$  is symmetric. Assume that  $b_1, b_2 \in B(\infty)$  satisfy the conditions:*

- (a) *one of  $G^{\text{up}}(b_1)^2$  and  $G^{\text{up}}(b_2)^2$  is a member of the upper global basis up to a power of  $q$ ,*
- (b)  *$G^{\text{up}}(b_1)$  and  $G^{\text{up}}(b_2)$   $q$ -commute.*

*Then their product  $G^{\text{up}}(b_1)G^{\text{up}}(b_2)$  is a member of the upper global basis of  $U_q^-(\mathfrak{g})$  up to a power of  $q$ .*

**3.2. Geometric results.** The result of this subsection (Theorem 3.6) is informed us by Peter McNamara.

*In this subsection, we assume further that the base field  $\mathbf{k}$  is a field of characteristic 0.*

**Theorem 3.6** ([20, Lemma 7.5]). *Assume that the base field  $\mathbf{k}$  is a field of characteristic 0. Assume that  $M \in R\text{-gmod}$  has a head  $q^c H$  with a self-dual simple module  $H$  and  $c \in \mathbb{Z}$ . Then we have the equality in the Grothendieck group  $K(R\text{-gmod})$*

$$[M] = q^c [H] + \sum_k q^{c_k} [S_k]$$

*with self-dual simple modules  $S_k$  and  $c_k > c$ .*

By duality, we obtain the following corollary.

**Corollary 3.7.** *Assume that the base field  $\mathbf{k}$  is a field of characteristic 0. Assume that  $M \in R\text{-gmod}$  has a socle  $q^c S$  with a self-dual simple module  $S$  and  $c \in \mathbb{Z}$ . Then we have the equality in  $K(R\text{-gmod})$*

$$[M] = q^c[S] + \sum_k q^{c_k}[S_k]$$

with self-dual simple modules  $S_k$  and  $c_k < c$ .

Applying this theorem to convolution products, we obtain the following corollary.

**Corollary 3.8.** *Assume that the base field  $\mathbf{k}$  is a field of characteristic 0. Let  $M$  and  $N$  be simple modules. We assume that one of them is real. Then we have the equalities in  $K(R\text{-gmod})$ :*

- (i)  $[M \circ N] = [M \diamond N] + \sum_k q^{c_k}[S_k]$   
with self-dual simple modules  $S_k$  and

$$c_k > -\tilde{\Lambda}(M, N) = (-\Lambda(M, N) - (\text{wt}(M), \text{wt}(N)))/2.$$

- (ii)  $[M \circ N] = [q^{\Lambda(N, M)} N \diamond M] + \sum_k q^{c_k}[S_k]$   
with self-dual simple modules  $S_k$  and  $c_k < (\Lambda(N, M) - (\text{wt}(N), \text{wt}(M)))/2$ .

Note that  $q^{\tilde{\Lambda}(M, N)} M \diamond N$  is self-dual by Lemma 2.7.

Theorem 3.1 and Theorem 3.6 solve affirmatively Conjecture 1 of Leclerc ([18]) in the symmetric generalized Cartan matrix case.

We obtain the following result which is a generalization of Lemma 2.24 in the characteristic-zero case.

**Corollary 3.9.** *Assume that the base field  $\mathbf{k}$  is a field of characteristic 0. Let  $M$  and  $N$  be simple modules. We assume that one of them is real. Write*

$$[M \circ N] = \sum_{k=1}^n q^{c_k}[S_k]$$

with self-dual simple modules  $S_k$  and  $c_k \in \mathbb{Z}$ . Then we have

$$\max \{c_k \mid 1 \leq k \leq n\} - \min \{c_k \mid 1 \leq k \leq n\} = \mathfrak{d}(M, N).$$

**3.3. Proof of Theorem 3.6.** Recall that the graded algebra  $R(\beta)$  ( $\beta \in \mathbb{Q}^+$ ) is geometrically realized as follows ([24]). There exist a reductive group  $G$  and a  $G$ -equivariant projective morphism  $f: X \rightarrow Y$  from a smooth algebraic  $G$ -variety  $X$  to an affine  $G$ -variety  $Y$  defined over the complex number field  $\mathbb{C}$  such that

$$R(\beta) \simeq \tilde{\text{End}}_{\text{D}_G^b(\mathbf{k}_Y)}(Rf_* \mathbf{k}_X) \quad \text{as a graded } \mathbf{k}\text{-algebra.}$$

Here,  $D_G^b(\mathbf{k}_Y)$  denotes the equivariant derived category of the  $G$ -variety  $Y$  with coefficient  $\mathbf{k}$ , and  $\tilde{\text{End}}_{D_G^b(\mathbf{k}_Y)}(K) = \tilde{\text{Hom}}_{D_G^b(\mathbf{k}_Y)}(K, K)$  with

$$\tilde{\text{Hom}}_{D_G^b(\mathbf{k}_Y)}(K, K') := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_G^b(\mathbf{k}_Y)}(K, K'[n]).$$

By the decomposition theorem ([1]), we have a decomposition

$$Rf_* \mathbf{k}_X \simeq \bigoplus_{a \in J} E_a \otimes \mathcal{F}_a$$

where  $\{\mathcal{F}_a\}_{a \in J}$  is a finite family of simple perverse sheaves on  $Y$  and  $E_a$  is a finite-dimensional graded  $\mathbf{k}$ -vector space such that

$$(3.1) \quad H^k(E_a) \simeq H^{-k}(E_a) \quad \text{for any } k \in \mathbb{Z}.$$

The last fact (3.1) follows from the hard Lefschetz theorem ([1]).

Set  $A_{a,b} = \tilde{\text{Hom}}_{D_G^b(\mathbf{k}_Y)}(\mathcal{F}_b, \mathcal{F}_a)$ . Then we have the multiplication morphisms

$$A_{a,b} \otimes A_{b,c} \rightarrow A_{a,c}$$

so that

$$A := \bigoplus_{a,b \in J} A_{a,b}$$

has a structure of  $\mathbb{Z}$ -graded algebra such that

$$A_{\leq 0} := \bigoplus_{n \leq 0} A_n = A_0 \simeq \mathbf{k}^J.$$

Hence the family of the isomorphism classes of simple objects (up to a grading shift) in  $A\text{-gmod}$  is  $\{\mathbf{k}_a\}_{a \in J}$ . Here,  $\mathbf{k}_a$  is the module obtained by the algebra homomorphism  $A \rightarrow A_{\leq 0} \simeq \mathbf{k}^J \rightarrow \mathbf{k}$ , where the last arrow is the  $a$ -th projection. Hence we have

$$K(A\text{-gmod}) \simeq \bigoplus_{a \in J} \mathbb{Z}[q^{\pm 1}][\mathbf{k}_a].$$

On the other hand, we have

$$R(\beta) \simeq \bigoplus_{a,b \in J} E_a \otimes A_{a,b} \otimes E_b^*.$$

Set

$$L := \bigoplus_{a,b \in J} E_a \otimes A_{a,b}.$$

Then,  $L$  is endowed with a natural structure of  $(\bigoplus_{a,b \in J} E_a \otimes A_{a,b} \otimes E_b^*, A)$ -bimodule. It is well-known that the functor  $M \mapsto L \otimes_A M$  gives a graded Morita-equivalence

$$\Phi: A\text{-gmod} \xrightarrow{\sim} R(\beta)\text{-gmod}.$$

Note that  $\Phi(\mathbf{k}_a) \simeq E_a$  and  $\{E_a\}_{a \in J}$  is the set of isomorphism classes of self-dual simple graded  $R(\beta)$ -modules by (3.1).

By the above observation, in order to prove the theorem, it is enough to show the corresponding statement for the graded ring  $A$ , which is obvious.  $\square$

#### 4. QUANTUM CLUSTER ALGEBRAS

In this section we recall the definition of skew-symmetric quantum cluster algebras following [2], [8, §8].

**4.1. Quantum seeds.** Fix a finite index set  $J = J_{\text{ex}} \sqcup J_{\text{fr}}$  with the decomposition into the set of exchangeable indices and the set of frozen indices. Let  $L = (\lambda_{ij})_{i,j \in J}$  be a skew-symmetric integer-valued  $J \times J$ -matrix.

**Definition 4.1.** We define  $\mathcal{P}(L)$  as the  $\mathbb{Z}[q^{\pm 1/2}]$ -algebra generated by a family of elements  $\{X_i\}_{i \in J}$  with the defining relations

$$(4.1) \quad X_i X_j = q^{\lambda_{ij}} X_j X_i \quad (i, j \in J).$$

We denote by  $\mathcal{F}(L)$  the skew field of fractions of  $\mathcal{P}(L)$ .

For  $\mathbf{a} = (a_i)_{i \in J} \in \mathbb{Z}^J$ , we define the element  $X^{\mathbf{a}}$  of  $\mathcal{F}(L)$  as

$$X^{\mathbf{a}} := q^{1/2 \sum_{i > j} a_i a_j \lambda_{ij}} \prod_{i \in J}^{\rightarrow} X_i^{a_i}.$$

Here we take a total order  $<$  on the set  $J$  and  $\prod_{i \in J}^{\rightarrow} X_i^{a_i} = X_{i_1}^{a_{i_1}} \cdots X_{i_r}^{a_{i_r}}$  where  $J = \{i_1, \dots, i_r\}$  with  $i_1 < \dots < i_r$ . Note that  $X^{\mathbf{a}}$  does not depend on the choice of a total order of  $J$ .

We have

$$(4.2) \quad X^{\mathbf{a}} X^{\mathbf{b}} = q^{1/2 \sum_{i,j \in J} a_i b_j \lambda_{ij}} X^{\mathbf{a} + \mathbf{b}}.$$

If  $\mathbf{a} \in \mathbb{Z}_{\geq 0}^J$ , then  $X^{\mathbf{a}}$  belongs to  $\mathcal{P}(L)$ .

It is well known that  $\{X^{\mathbf{a}}\}_{\mathbf{a} \in \mathbb{Z}_{\geq 0}^J}$  is a basis of  $\mathcal{P}(L)$  as a  $\mathbb{Z}[q^{\pm 1/2}]$ -module.

Let  $A$  be a  $\mathbb{Z}[q^{\pm 1/2}]$ -algebra. We say that a family  $\{x_i\}_{i \in J}$  of elements of  $A$  is *L-commuting* if it satisfies  $x_i x_j = q^{\lambda_{ij}} x_j x_i$  for any  $i, j \in J$ . In such a case we can define  $x^{\mathbf{a}}$  for any  $\mathbf{a} \in \mathbb{Z}_{\geq 0}^J$ . We say that an *L-commuting* family  $\{x_i\}_{i \in J}$  is *algebraically independent* if the algebra map  $\mathcal{P}(L) \rightarrow A$  given by  $X_i \mapsto x_i$  is injective.

Let  $\tilde{B} = (b_{ij})_{(i,j) \in J \times J_{\text{ex}}}$  be an integer-valued  $J \times J_{\text{ex}}$ -matrix. We assume that the *principal part*  $B := (b_{ij})_{i,j \in J_{\text{ex}}}$  of  $\tilde{B}$  is skew-symmetric. We say that the pair  $(L, \tilde{B})$  is *compatible*, if there exists a positive integer  $d$  such that

$$(4.3) \quad \sum_{k \in J} \lambda_{ik} b_{kj} = \delta_{ij} d \quad (i \in J, j \in J_{\text{ex}}).$$

Let  $(L, B)$  be a compatible pair and  $A$  a  $\mathbb{Z}[q^{\pm 1/2}]$ -algebra. We say that  $\mathcal{S} = (\{x_i\}_{i \in J}, L, \tilde{B})$  is a *quantum seed* in  $A$  if  $\{x_i\}_{i \in J}$  is an algebraically independent  $L$ -commuting family of elements of  $A$ .

The set  $\{x_i\}_{i \in J}$  is called the *cluster* of  $\mathcal{S}$  and its elements *cluster variables*. The cluster variables  $x_i$  ( $i \in J_{\text{fr}}$ ) are called *frozen variables*. The elements  $x^{\mathbf{a}}$  ( $\mathbf{a} \in \mathbb{Z}_{\geq 0}^J$ ) are called *quantum cluster monomials*.

**4.2. Mutation.** For  $k \in J_{\text{ex}}$ , we define a  $J \times J$ -matrix  $E = (e_{ij})_{i,j \in J}$  and a  $J_{\text{ex}} \times J_{\text{ex}}$ -matrix  $F = (f_{ij})_{i,j \in J_{\text{ex}}}$  as follows:

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, -b_{ik}) & \text{if } i \neq j = k, \end{cases} \quad f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, b_{kj}) & \text{if } i = k \neq j. \end{cases}$$

The *mutation*  $\mu_k(L, \tilde{B}) := (\mu_k(L), \mu_k(\tilde{B}))$  of a compatible pair  $(L, \tilde{B})$  in direction  $k$  is given by

$$\mu_k(L) := (E^T) L E, \quad \mu_k(\tilde{B}) := E \tilde{B} F.$$

Then the pair  $(\mu_k(L), \mu_k(\tilde{B}))$  is also compatible with the same integer  $d$  as in the case of  $(L, \tilde{B})$  ([2]).

Note that for each  $k \in J_{\text{ex}}$ , we have

$$(4.4) \quad \mu_k(\tilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + (-1)^{\delta(b_{ik} < 0)} \max(b_{ik} b_{kj}, 0) & \text{otherwise,} \end{cases}$$

and

$$\mu_k(L)_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\lambda_{kj} + \sum_{t \in J} \max(0, -b_{tk}) \lambda_{tj} & \text{if } i = k, j \neq k, \\ -\lambda_{ik} + \sum_{t \in J} \max(0, -b_{tk}) \lambda_{it} & \text{if } i \neq k, j = k, \\ \lambda_{ij} & \text{otherwise.} \end{cases}$$

Note that we have

$$\sum_{t \in J} \max(0, -b_{tk}) \lambda_{it} = \sum_{t \in J} \max(0, b_{tk}) \lambda_{it}$$

for  $i \in J$  with  $i \neq k$ , since  $(L, \tilde{B})$  is compatible.

We define

$$(4.5) \quad a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \quad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k. \end{cases}$$

and set  $\mathbf{a}' := (a'_i)_{i \in J}$  and  $\mathbf{a}'' := (a''_i)_{i \in J}$ .

Let  $A$  be a  $\mathbb{Z}[q^{\pm 1/2}]$ -algebra contained in a skew field  $K$ . Let  $\mathcal{S} = (\{x_i\}_{i \in J}, L, \tilde{B})$  be a quantum seed in  $A$ . Define the elements  $\mu_k(x)_i$  of  $K$  by

$$(4.6) \quad \mu_k(x)_i := \begin{cases} x^{\mathbf{a}'} + x^{\mathbf{a}''}, & \text{if } i = k, \\ x_i & \text{if } i \neq k. \end{cases}$$

Then  $\{\mu_k(x)_i\}$  is an algebraically independent  $\mu_k(L)$ -commuting family in  $K$ . We call

$$\mu_k(\mathcal{S}) := (\{\mu_k(x)_i\}_{i \in J}, \mu_k(L), \mu_k(\tilde{B}))$$

the *mutation of  $\mathcal{S}$  in direction  $k$* . It becomes a new quantum seed in  $K$ .

**Definition 4.2.** Let  $\mathcal{S} = (\{x_i\}_{i \in J}, L, \tilde{B})$  be a quantum seed in  $A$ . The quantum cluster algebra  $\mathcal{A}_{q^{1/2}}(\mathcal{S})$  associated to the quantum seed  $\mathcal{S}$  is the  $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of the skew field  $K$  generated by all the quantum cluster variables in the quantum seeds obtained from  $\mathcal{S}$  by any sequence of mutations.

We call  $\mathcal{S}$  the *initial quantum seed* of the quantum cluster algebra  $\mathcal{A}_{q^{1/2}}(\mathcal{S})$ .

## 5. MONOIDAL CATEGORIFICATION

Throughout this section, fix  $J = J_{\text{ex}} \sqcup J_{\text{fr}}$ , and a base field  $\mathbf{k}$ .

Let  $\mathcal{C}$  be a  $\mathbf{k}$ -linear abelian monoidal category. For the definition of monoidal category, see [11, Appendix A.1]. Note that in [11], it was called *tensor category*. A  $\mathbf{k}$ -linear abelian monoidal category is a  $\mathbf{k}$ -linear monoidal category such that it is abelian and the tensor functor  $\otimes$  is  $\mathbf{k}$ -bilinear and exact.

We assume further the following conditions on  $\mathcal{C}$

$$(5.1) \quad \begin{cases} \text{(i) Any object of } \mathcal{C} \text{ has a finite length,} \\ \text{(ii) } \mathbf{k} \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(S, S) \text{ for any simple object } S \text{ of } \mathcal{C}. \end{cases}$$

### 5.1. Ungraded cases.

**Definition 5.1.** Let  $\mathcal{S} = (\{M_i\}_{i \in J}, \tilde{B})$  be a pair of a family  $\{M_i\}_{i \in J}$  of simple objects in  $\mathcal{C}$  and an integer-valued  $J \times J_{\text{ex}}$ -matrix  $\tilde{B} = (b_{ij})_{(i,j) \in J \times J_{\text{ex}}}$  whose principal part is skew-symmetric. We call  $\mathcal{S}$  a *monoidal seed* in  $\mathcal{C}$  if

- (i)  $M_i \otimes M_j \simeq M_j \otimes M_i$  for any  $i, j \in J$ , and
- (ii)  $\bigotimes_{i \in J} M_i^{\otimes a_i}$  is simple for any  $(a_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$ .

**Definition 5.2.** For  $k \in J_{\text{ex}}$ , we say that a monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, \tilde{B})$  admits a mutation in direction  $k$  if there exists a simple object  $M'_k \in \mathcal{C}$  such that



(a) *there exist exact sequences in  $\mathcal{C}$*

$$(5.2) \quad 0 \rightarrow \bigotimes_{b_{ik} > 0} M_i^{\otimes b_{ik}} \rightarrow M_k \otimes M'_k \rightarrow \bigotimes_{b_{ik} < 0} M_i^{\otimes (-b_{ik})} \rightarrow 0,$$

$$(5.3) \quad 0 \rightarrow \bigotimes_{b_{ik} < 0} M_i^{\otimes (-b_{ik})} \rightarrow M'_k \otimes M_k \rightarrow \bigotimes_{b_{ik} > 0} M_i^{\otimes b_{ik}} \rightarrow 0.$$

(b) *the pair  $\mu_k(\mathcal{S}) := (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(\tilde{B}))$  is a monoidal seed in  $\mathcal{C}$ .*

Recall that a cluster algebra  $A$  with an initial seed  $(\{x_i\}_{i \in J}, \tilde{B})$  is the  $\mathbb{Z}$ -subalgebra of  $\mathbb{Q}(x_i | i \in J)$  generated by all the cluster variables in the seeds obtained from  $(\{x_i\}_{i \in J}, \tilde{B})$  by any sequence of mutations. Here, the mutation  $x'_k$  of a cluster variable  $x_k$  ( $k \in J_{\text{ex}}$ ) is given by

$$x'_k = \frac{\prod_{b_{ik} \geq 0} x_i^{b_{ik}} + \prod_{b_{ik} \leq 0} x_i^{-b_{ik}}}{x_k},$$

and the mutation of  $\tilde{B}$  is given in (4.4).

**Definition 5.3.** *A  $\mathbf{k}$ -linear abelian monoidal category  $\mathcal{C}$  with (5.1) is called a monoidal categorification of a cluster algebra  $A$  if*

- (a) *the Grothendieck ring  $K(\mathcal{C})$  is isomorphic to  $A$ ,*
- (b) *there exists a monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, \tilde{B})$  in  $\mathcal{C}$  such that  $[\mathcal{S}] := (\{[M_i]\}_{i \in J}, \tilde{B})$  is the initial seed of  $A$  and  $\mathcal{S}$  admits successive mutations in all directions.*

Note that if  $\mathcal{C}$  is a monoidal categorification of  $A$ , then every seed in  $A$  is of the form  $(\{[M_i]\}_{i \in J}, \tilde{B})$  for some monoidal seed  $(\{M_i\}_{i \in J}, \tilde{B})$ . In particular, all the cluster monomials in  $A$  are the classes of real simple objects in  $\mathcal{C}$ .

**5.2. Graded cases.** Let  $\mathbf{Q}$  be a free abelian group equipped with a symmetric bilinear form

$$(\ , \ ) : \mathbf{Q} \times \mathbf{Q} \rightarrow \mathbb{Z} \quad \text{such that } (\beta, \beta) \in 2\mathbb{Z} \text{ for all } \beta \in \mathbf{Q}.$$

We consider a  $\mathbf{k}$ -linear abelian monoidal category  $\mathcal{C}$  satisfying (5.1) and the following conditions:

$$(5.4) \quad \left\{ \begin{array}{l} \text{(i) We have a direct sum decomposition } \mathcal{C} = \bigoplus_{\beta \in \mathbb{Q}} \mathcal{C}_\beta \text{ such that the} \\ \text{tensor product } \otimes \text{ sends } \mathcal{C}_\beta \times \mathcal{C}_\gamma \text{ to } \mathcal{C}_{\beta+\gamma} \text{ for every } \beta, \gamma \in \mathbb{Q}. \\ \text{(ii) There exists an object } Q \in \mathcal{C}_0 \text{ satisfying} \\ \quad \text{(a) there is an isomorphism} \\ \qquad R_Q(X) : Q \otimes X \xrightarrow{\sim} X \otimes Q \\ \qquad \text{functorial in } X \in \mathcal{C} \text{ such that} \\ \qquad \qquad \qquad \begin{array}{ccccc} & & R_Q(X \otimes Y) & & \\ & \nearrow & & \searrow & \\ Q \otimes X \otimes Y & \xrightarrow{R_Q(X)} & X \otimes Q \otimes Y & \xrightarrow{R_Q(Y)} & X \otimes Y \otimes Q \end{array} \\ \qquad \text{commutes for any } X, Y \in \mathcal{C}, \\ \quad \text{(b) the functor } X \mapsto Q \otimes X \text{ is an equivalence of categories.} \\ \text{(iii) for any } M, N \in \mathcal{C}, \text{ we have } \text{Hom}_{\mathcal{C}}(M, Q^{\otimes n} \otimes N) = 0 \text{ except} \\ \qquad \text{finitely many integers } n. \end{array} \right.$$

We denote by  $q$  the auto-equivalence  $Q \otimes \bullet$  of  $\mathcal{C}$ , and call it the *grading shift functor*.

In such a case the Grothendieck group  $K(\mathcal{C})$  is a  $\mathbb{Q}$ -graded  $\mathbb{Z}[q^{\pm 1}]$ -algebra:  $K(\mathcal{C}) = \bigoplus_{\beta \in \mathbb{Q}} K(\mathcal{C})_\beta$  where  $K(\mathcal{C})_\beta = K(\mathcal{C}_\beta)$ . Moreover we have

$$K(\mathcal{C}) = \bigoplus_S \mathbb{Z}[q^{\pm 1}][S]$$

where  $S$  ranges over the set of equivalence classes of simple modules  $S$ . Here, two simple modules  $S$  and  $S'$  are equivalent if  $q^n S \simeq S'$  for some  $n \in \mathbb{Z}$ .

For  $M \in \mathcal{C}_\beta$ , we sometimes write  $\beta = \text{wt}(M)$  and call it the *weight* of  $M$ . Similarly, for  $x \in \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C}_\beta)$ , we write  $\beta = \text{wt}(x)$  and call it the *weight* of  $x$ .

**Definition 5.4.** We call a quadruple  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  a quantum monoidal seed in  $\mathcal{C}$  if it satisfies the following conditions:

- (i)  $\tilde{B} = (b_{ij})_{i \in J, j \in J_{\text{ex}}}$  is an integer-valued  $J \times J_{\text{ex}}$ -matrix whose principal part is skew-symmetric,
- (ii)  $L = (\lambda_{ij \in J})$  is an integer-valued skew-symmetric  $J \times J$ -matrix,
- (iii)  $D = \{d_i\}_{i \in J}$  is a family of elements in  $\mathbb{Q}$ ,
- (iv)  $\{M_i\}_{i \in J}$  is a family of simple objects such that  $M_i \in \mathcal{C}_{d_i}$  for any  $i \in J$ ,
- (v)  $M_i \otimes M_j \simeq q^{\lambda_{ij}} M_j \otimes M_i$  for all  $i, j \in J$ ,
- (vi)  $M_{i_1} \otimes \cdots \otimes M_{i_t}$  is simple for any sequence  $(i_1, \dots, i_t)$  in  $J$ ,
- (vii) The pair  $(L, \tilde{B})$  is compatible in the sense of (4.3) with  $d = 2$ ,
- (viii)  $\lambda_{ij} - (d_i, d_j) \in 2\mathbb{Z}$  for all  $i, j \in J$ ,

$$(ix) \sum_{i \in J} b_{ik} d_i = 0 \text{ for all } k \in J_{\text{ex}}.$$

Let  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  be a quantum monoidal seed. For any  $X \in \mathcal{C}_\beta$  and  $Y \in \mathcal{C}_\gamma$  such that  $X \circ Y \simeq q^c Y \circ X$  and  $c + (\beta, \gamma) \in 2\mathbb{Z}$ , we set

$$(5.5) \quad \tilde{\Lambda}(X, Y) = \frac{1}{2}(-c + (\beta, \gamma)) \in \mathbb{Z}$$

and

$$(5.6) \quad X \odot Y := q^{\tilde{\Lambda}(X, Y)} X \otimes Y \simeq q^{\tilde{\Lambda}(Y, X)} Y \otimes X.$$

Then  $X \odot Y \simeq Y \odot X$ . For any sequence  $(i_1, \dots, i_\ell)$  in  $J$ , we define

$$\bigodot_{s=1}^{\ell} M_{i_s} := (\dots ((M_{i_1} \odot M_{i_2}) \odot M_{i_3}) \dots) \odot M_{i_\ell}.$$

Then we have

$$\bigodot_{s=1}^{\ell} M_{i_s} = q^{\frac{1}{2} \sum_{1 \leq u < v \leq \ell} (-\lambda_{i_u i_v} + (d_{i_u}, d_{i_v}))} M_{i_1} \otimes \dots \otimes M_{i_\ell}.$$

For any  $w \in \mathfrak{S}_\ell$ , we have

$$\bigodot_{s=1}^{\ell} M_{i_{w(s)}} \simeq \bigodot_{s=1}^{\ell} M_{i_s}$$

Hence for any subset  $A$  of  $J$  and any set of non-negative integers  $\{m_a\}_{a \in A}$ , we can define  $\bigodot_{a \in A} M_a^{\odot m_a}$ .

For  $(a_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$  and  $(b_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$ , we have

$$\left( \bigodot_{i \in J} M_i^{\odot a_i} \right) \odot \left( \bigodot_{i \in J} M_i^{\odot b_i} \right) \simeq \bigodot_{i \in J} M_i^{\odot (a_i + b_i)}.$$

Let  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  be a quantum monoidal seed. When the  $L$ -commuting family  $\{[M_i]\}_{i \in J}$  of elements of  $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  is algebraically independent, we shall define a quantum seed  $[\mathcal{S}]$  in  $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  by

$$(5.7) \quad [\mathcal{S}] = (\{q^{-(d_i, d_i)/4} [M_i]\}_{i \in J}, L, \tilde{B}).$$

Set

$$(5.8) \quad X_i = q^{-(d_i, d_i)/4} [M_i].$$

Then for any  $\mathbf{a} = (a_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$ , we have

$$\left[ \bigodot_{i \in J} M_i^{\odot a_i} \right] = q^{(\mu, \mu)/4} X^{\mathbf{a}}$$

where  $\mu = \text{wt}(\bigodot_{i \in J} M_i^{\odot a_i}) = \text{wt}(X^{\mathbf{a}}) = \sum_{i \in J} a_i d_i$ .

For a given  $k \in J_{\text{ex}}$ , we define the *mutation*  $\mu_k(D) \in \mathbb{Q}^J$  of  $D$  in direction  $k$  with respect to  $\tilde{B}$  by

$$\mu_k(D)_i = d_i \ (i \neq k), \quad \mu_k(D)_k = -d_k + \sum_{b_{ik} > 0} b_{ik} d_i.$$

Note that

$$\mu_k(\mu_k(D)) = D.$$

Note also that  $(\mu_k(L), \mu_k(\tilde{B}), \mu_k(D))$  satisfies conditions (viii) and (ix) in Definition 5.4 for any  $k \in J_{\text{ex}}$ .

Note that

**Lemma 5.5.** *Set  $X'_k = \mu_k(X)_k$ , the mutation of  $X_k$  as in (4.6). Set  $\zeta = \text{wt}(X'_k) = -d_k + \sum_{b_{ik} > 0} b_{ik} d_i$ . Then we have*

$$(5.9) \quad \begin{aligned} q^{m_k} [M_k] q^{(\zeta, \zeta)/4} X'_k &= q \left[ \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \right] + \left[ \bigodot_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \right], \\ q^{m'_k} q^{(\zeta, \zeta)/4} X'_k [M_k] &= \left[ \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \right] + q \left[ \bigodot_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \right], \end{aligned}$$

where

$$(5.10) \quad \begin{cases} m_k = \frac{1}{2}(d_k, \zeta) + \frac{1}{2} \sum_{b_{ik} < 0} \lambda_{ki} b_{ik}, \\ m'_k = \frac{1}{2}(d_k, \zeta) + \frac{1}{2} \sum_{b_{ik} > 0} \lambda_{ki} b_{ik}. \end{cases}$$

*Proof.* By (4.2), we have

$$X_k X^{\mathbf{a}} = q^{\frac{1}{2} \sum_{i \in J} a_i \lambda_{ki}} X^{\mathbf{e}_k + \mathbf{a}} \quad \text{for } \mathbf{a} = (a_i)_{i \in J} \in \mathbb{Z}^J \text{ and } (\mathbf{e}_k)_i = \delta_{ik} \ (i \in J).$$

Let  $\mathbf{a}'$  and  $\mathbf{a}''$  be as in (4.5). Because

$$\sum_{i \in J} a'_i \lambda_{ki} - \sum_{i \in J} a''_i \lambda_{ki} = \sum_{b_{ik} > 0} b_{ik} \lambda_{ki} - \sum_{b_{ik} < 0} (-b_{ik}) \lambda_{ki} = \sum_{i \in J} b_{ik} \lambda_{ki} = 2,$$

we have

$$X_k X'_k = X_k (X^{\mathbf{a}'} + X^{\mathbf{a}''}) = q^{\frac{1}{2} \sum_i a''_i \lambda_{ki}} (q X^{\mathbf{e}_k + \mathbf{a}'} + X^{\mathbf{e}_k + \mathbf{a}''}).$$

Note that  $\text{wt}(X^{\mathbf{e}_k + \mathbf{a}'}) = \text{wt}(X^{\mathbf{e}_k + \mathbf{a}''}) = d_k + \zeta$ . It follows that

$$\begin{aligned} m_k &= -\frac{1}{4}((d_k, d_k) + (\zeta, \zeta)) - \frac{1}{2} \sum_{i \in J} a''_i \lambda_{ki} + \frac{1}{4}(\zeta + d_k, \zeta + d_k) \\ &= \frac{1}{2}(d_k, \zeta) + \frac{1}{2} \sum_{b_{ik} < 0} b_{ik} \lambda_{ki}. \end{aligned}$$

One can calculate  $m'_k$  in a similar way.  $\square$

**Definition 5.6.** We say that a quantum monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  admits a mutation in direction  $k \in J_{\text{ex}}$  if there exists a simple object  $M'_k \in \mathcal{C}_{\mu_k(D)_k}$  such that

(a) there exist exact sequences in  $\mathcal{C}$

$$(5.11) \quad 0 \rightarrow q \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \rightarrow q^{m_k} M_k \otimes M'_k \rightarrow \bigodot_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \rightarrow 0,$$

$$(5.12) \quad 0 \rightarrow q \bigodot_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \rightarrow q^{m'_k} M'_k \otimes M_k \rightarrow \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \rightarrow 0,$$

where  $m_k$  and  $m'_k$  are as in (5.10).

(b)  $\mu_k(\mathcal{S}) := (\{M_i\}_{i \neq k} \sqcup \{M'_k\}, \mu_k(L), \mu_k(\tilde{B}), \mu_k(D))$  is a quantum monoidal seed in  $\mathcal{C}$ . We call  $\mu_k(\mathcal{S})$  the mutation of  $\mathcal{S}$  in direction  $k$ .

By Lemma 5.5, the following lemma is obvious.

**Lemma 5.7.** Let  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  be a quantum monoidal seed which admits a mutation in direction  $k \in J_{\text{ex}}$ . Then we have

$$[\mu_k(\mathcal{S})] = \mu_k([\mathcal{S}]).$$

**Definition 5.8.** Assume that a  $\mathbf{k}$ -linear abelian monoidal category  $\mathcal{C}$  satisfies (5.1) and (5.4). The category  $\mathcal{C}$  is called a monoidal categorification of a quantum cluster algebra  $A$  over  $\mathbb{Z}[q^{\pm 1/2}]$  if

- (i) the Grothendieck ring  $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  is isomorphic to  $A$ ,
- (ii) there exists a quantum monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$  in  $\mathcal{C}$  such that  $[\mathcal{S}] := (\{q^{-(d_i, d_i)/4} [M_i]\}_{i \in J}, L, \tilde{B})$  is a quantum seed of  $A$ ,
- (iii)  $\mathcal{S}$  admits successive mutations in all the directions.

Note that if  $\mathcal{C}$  is a monoidal categorification of a quantum cluster algebra  $A$ , then any quantum seed in  $A$  obtained by mutations from the initial quantum seed is of the form  $(\{q^{-(d_i, d_i)/4} [M_i]\}_{i \in J}, L, \tilde{B})$  for some quantum monoidal seed  $(\{M_i\}_{i \in J}, L, \tilde{B}, D)$ . In particular, all the quantum cluster monomials in  $A$  are the classes of real simple objects in  $\mathcal{C}$  up to a power of  $q^{1/2}$ .

## 6. MONOIDAL CATEGORIFICATION VIA MODULES OVER KLR ALGEBRAS

Let  $R$  be a symmetric KLR algebra over a base field  $\mathbf{k}$ .

From now on, we focus on the case when  $\mathcal{C}$  is a full subcategory of  $R$ -gmod stable under taking convolution products, subquotients, extensions and grading shift. In particular, we have

$$\mathcal{C} = \bigoplus_{\beta \in \mathbf{Q}^-} \mathcal{C}_\beta, \quad \text{where } \mathcal{C}_\beta := \mathcal{C} \cap R(-\beta)\text{-gmod},$$

and we have the grading shift functor  $q$  on  $\mathcal{C}$ . Hence we have

$$K(\mathcal{C}_\beta) \subset U_q^-(\mathfrak{g})_\beta,$$

and  $K(\mathcal{C})$  has a  $\mathbb{Z}[q^{\pm 1}]$ -basis consisting of the isomorphism classes of self-dual simple modules.

**Definition 6.1.** A pair  $(\{M_i\}_{i \in J}, \tilde{B})$  is called admissible if

- (a)  $\{M_i\}_{i \in J}$  is a family of real simple self-dual objects of  $\mathcal{C}$  which commute with each other,
- (b)  $\tilde{B}$  is an integer-valued  $J \times J_{\text{ex}}$ -matrix with skew-symmetric principal part,
- (c) for each  $k \in J_{\text{ex}}$ , there exists a self-dual simple object  $M'_k$  of  $\mathcal{C}$  such that there is an exact sequence in  $\mathcal{C}$

$$(6.1) \quad 0 \rightarrow q \bigoplus_{b_{ik} > 0} M_i^{\odot b_{ik}} \rightarrow q^{\tilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \rightarrow \bigoplus_{b_{ik} < 0} M_i^{\odot (-b_{ik})} \rightarrow 0,$$

and  $M'_k$  commutes with  $M_i$  for any  $i \neq k$ .

Note that  $M'_k$  is uniquely determined by  $k$  and  $(\{M_i\}_{i \in J}, \tilde{B})$ . Indeed, it follows from  $q^{\tilde{\Lambda}(M_k, M'_k)} M_k \diamond M'_k \simeq \bigoplus_{b_{ik} < 0} M_i^{\odot (-b_{ik})}$  and [12, Corollary 3.7]. Note also that if there is an epimorphism  $q^m M_k \circ M'_k \twoheadrightarrow \bigoplus_{b_{ik} < 0} M_i^{\odot (-b_{ik})}$  for some  $m \in \mathbb{Z}$ , then  $m$  should coincide with  $\tilde{\Lambda}(M_k, M'_k)$  by Lemma 2.7 and Lemma 2.15.

For an admissible pair  $(\{M_i\}_{i \in J}, \tilde{B})$ , let  $\Lambda = (\Lambda_{ij})_{i, j \in J}$  be the skew-symmetric matrix given by  $\Lambda_{ij} = \Lambda(M_i, M_j)$ . and let  $D = \{d_i\}_{i \in J}$  be the family of elements of  $\mathbb{Q}^-$  given by  $d_i = \text{wt}(M_i)$ .

Now we can simplify the conditions in Definition 5.4 and Definition 5.6 as follows.

**Proposition 6.2.** Let  $(\{M_i\}_{i \in J}, \tilde{B})$  be an admissible pair in  $\mathcal{C}$ , and let  $M'_k$  ( $k \in J_{\text{ex}}$ ) be as in Definition 6.1. Then we have the following properties.

- (i) The quadruple  $\mathcal{S} := (\{M_i\}_{i \in J}, -\Lambda, \tilde{B}, D)$  is a quantum monoidal seed in  $\mathcal{C}$ .
- (ii) The self-dual simple object  $M'_k$  is real for every  $k \in J_{\text{ex}}$ .
- (iii) The quantum monoidal seed  $\mathcal{S}$  admits a mutation in each direction  $k \in J_{\text{ex}}$ .
- (iv)  $M_k$  and  $M'_k$  is simply-linked for any  $k \in J_{\text{ex}}$  (i.e.,  $\mathfrak{d}(M_k, M'_k) = 1$ ).
- (v) For any  $j \in J$  and  $k \in J_{\text{ex}}$ , we have

$$(6.2) \quad \begin{aligned} \Lambda(M_j, M'_k) &= -\Lambda(M_j, M_k) - \sum_{b_{ik} < 0} \Lambda(M_j, M_i) b_{ik}, \\ \Lambda(M'_k, M_j) &= -\Lambda(M_k, M_j) + \sum_{b_{ik} > 0} \Lambda(M_i, M_j) b_{ik}. \end{aligned}$$

*Proof.* (iv) follows from the exact sequence (6.1) and Corollary 2.24.

(ii) follows from the exact sequence (6.1) by applying Corollary 2.22 to the case

$$M = M_k, \quad N = M'_k, \quad X = q \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}}, \quad \text{and} \quad Y = \bigodot_{b_{ik} < 0} M_i^{\odot(-b_{ik})}.$$

(v) follows from

$$\begin{aligned} \Lambda(M_j, M_k) + \Lambda(M_j, M'_k) &= \Lambda(M_j, M_k \diamond M'_k) = \Lambda(M_j, \bigodot_{b_{ik} < 0} M_i^{\odot(-b_{ik})}) \\ &= \sum_{b_{ik} < 0} \Lambda(M_j, M_i)(-b_{ik}) \end{aligned}$$

and

$$\begin{aligned} \Lambda(M_k, M_j) + \Lambda(M'_k, M_j) &= \Lambda(M'_k \diamond M_k, M_j) = \Lambda(\bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}}, M_j) \\ &= \sum_{b_{ik} > 0} \Lambda(M_i, M_j)b_{ik}. \end{aligned}$$

Let us show (i). Conditions (i)–(v) in Definition 5.4 are satisfied by the construction. Condition (vi) follows from Proposition 2.13 and the fact that  $M_i$  is real simple for every  $i \in J$ . Condition (viii) is nothing but Lemma 2.5. Condition (ix) follows easily from the fact that the weights of the first and the last terms in the exact sequence (6.1) coincide.

Let us show condition (vii) in Definition 5.4. By (6.2) and (iv) of this proposition, we have

$$\begin{aligned} 2\delta_{jk} &= 2\mathfrak{d}(M_j, M'_k) = -2\mathfrak{d}(M_j, M_k) - \sum_{b_{ik} < 0} \Lambda(M_j, M_i)b_{ik} + \sum_{b_{ik} > 0} \Lambda(M_i, M_j)b_{ik} \\ &= - \sum_{b_{ik} < 0} \Lambda(M_j, M_i)b_{ik} - \sum_{b_{ik} > 0} \Lambda(M_j, M_i)b_{ik} = - \sum_{i \in J} \Lambda(M_j, M_i)b_{ik}. \end{aligned}$$

for  $k \in J_{\text{ex}}$  and  $j \in J$ .

Thus we have shown that  $\mathcal{S}$  is a quantum monoidal seed in  $\mathcal{C}$ .

Let us show (iii). Let  $k \in J_{\text{ex}}$ . The exact sequence (5.11) follows from (6.1) and the equality

$$(6.3) \quad \tilde{\Lambda}(M_k, M'_k) = \frac{1}{2}((\text{wt}(M_k, M'_k) - \sum_{b_{ik} < 0} \Lambda(M_k, M_i)b_{ik})) = m_k$$

which is an immediate consequence of (6.2).

Similarly, taking the dual of the exact sequence (6.1), we obtain an exact sequence

$$0 \rightarrow \bigodot_{b_{ik} < 0} M_i^{\odot(-b_{ik})} \rightarrow q^{-\tilde{\Lambda}(M_k, M'_k) + (\text{wt } M_k, \text{wt } M'_k)} M'_k \circ M_k \rightarrow q^{-1} \bigodot_{b_{ik} > 0} M_i^{\odot b_{ik}} \rightarrow 0,$$



which gives the exact sequence (5.12).

It remains to prove that  $\mu_k(\mathcal{S}) := (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(-\Lambda), \mu_k(\tilde{B}), \mu_k(D))$  is a quantum monoidal seed in  $\mathcal{C}$  for any  $k \in J_{\text{ex}}$ .

We see easily that  $\mu_k(\mathcal{S})$  satisfies the conditions (i)–(iv) and (vii)–(ix) in Definition 5.4.

For condition (v), it is enough to show that for  $i, j \in J$  we have

$$\mu_k(-\Lambda)_{ij} = -\Lambda(\mu_k(M)_i, \mu_k(M)_j),$$

where  $\mu_k(M)_i = M_i$  for  $i \neq k$  and  $\mu_k(M)_k = M'_k$ . In the case  $i \neq k$  and  $j \neq k$ , we have

$$\mu_k(-\Lambda)_{ij} = -\Lambda(M_i, M_j) = -\Lambda(\mu_k(M)_i, \mu_k(M)_j).$$

The other cases follow from (6.2).

Condition (vi) in Definition 5.4 for  $\mu_k(\mathcal{S})$  follows from Proposition 2.13 and the fact that  $\{\mu_k(M)_i\}_{i \in J}$  is a commuting family of real simple modules.  $\square$

Now we are ready to give our main theorem.

**Theorem 6.3.** *Let  $(\{M_i\}_{i \in K}, \tilde{B})$  be an admissible pair in  $\mathcal{C}$  and set*

$$\mathcal{S} = (\{M_i\}_{i \in J}, -\Lambda, \tilde{B}, D)$$

*as in Proposition 6.2. We set  $[\mathcal{S}] := (\{q^{-\frac{1}{4}(\text{wt}(M_i), \text{wt}(M_i))}[M_i]\}_{i \in J}, -\Lambda, \tilde{B}, D)$ . We assume further*

$$(6.4) \quad \text{The } \mathbb{Q}(q^{1/2})\text{-algebra } \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C}) \text{ is isomorphic to } \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{A}_{q^{1/2}}([\mathcal{S}]).$$

*Then, for each  $x \in J_{\text{ex}}$ , the pair  $(\{\mu_x(M)_i\}_{i \in J}, \mu_x(\tilde{B}))$  is admissible in  $\mathcal{C}$ .*

*Proof.* In Proposition 6.2, we have already proved the conditions (a) and (b) in Definition 6.1 for  $(\{\mu_x(M)_i\}_{i \in J}, \mu_x(\tilde{B}))$ . Let us show (c). Set  $N_i := \mu_x(M)_i$  and  $b'_{ij} := \mu_x(\tilde{B})_{ij}$  for  $i \in J$  and  $j \in J_{\text{ex}}$ . It is enough to show that, for any  $y \in J_{\text{ex}}$ , there exists a self-dual simple module  $M''_y \in \mathcal{C}$  such that there is a short exact sequence

$$(6.5) \quad 0 \longrightarrow q \bigodot_{b'_{iy} > 0} N_i^{\odot b'_{iy}} \longrightarrow q^{\tilde{\Lambda}(N_y, M''_y)} N_y \circ M''_y \longrightarrow \bigodot_{b'_{iy} < 0} N_i^{\odot (-b'_{iy})} \longrightarrow 0,$$

and we have

$$\mathfrak{d}(N_i, M''_y) = 0 \quad \text{for } i \neq y.$$

If  $x = y$ , then  $b'_{iy} = -b_{ix}$  and hence  $M''_y = M_x$  satisfies the desired condition.

Assume that  $x \neq y$  and  $b_{xy} = 0$ . Then  $b'_{iy} = b_{iy}$  for any  $i$  and  $N_i = M_i$  for any  $i \neq x$ . Hence  $M''_y = \mu_y(M)_y$  satisfies the desired condition.

We will show the assertion in the case  $b_{xy} > 0$ . We omit the proof of the case  $b_{xy} < 0$  because it can be shown in a similar way.

Set

$$\begin{aligned}
M'_x &:= \mu_x(M)_x, & M'_y &:= \mu_y(M)_y, \\
C &:= \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}, & S &:= \bigodot_{b_{ix}<0, i \neq y} M_i^{\odot -b_{ix}}, \\
P &:= \bigodot_{b_{iy}>0, i \neq x} M_i^{\odot b_{iy}}, & Q &= \bigodot_{b'_{iy}<0, i \neq x} M_i^{\odot -b'_{iy}}, \\
A &:= \bigodot_{b'_{iy} \leq 0, b_{ix}>0} M_i^{\odot b_{ix} b_{xy}} \bigodot_{b_{iy}<0, b'_{iy}>0, b_{ix}>0} M_i^{\odot -b_{iy}} \\
&\simeq \bigodot_{b_{iy}<0, b_{ix}>0} M_i^{\odot \min(b_{ix} b_{xy}, -b_{iy})}, \\
B &:= \bigodot_{b_{iy}>0, b_{ix}>0} M_i^{\odot b_{ix} b_{xy}} \bigodot_{b'_{iy}>0, b_{iy}<0, b_{ix}>0} M_i^{\odot b'_{iy}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
Q \odot A &\simeq \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}} \quad \text{and} \\
A \odot B &\simeq C^{\odot b_{xy}}.
\end{aligned}$$

Set

$$L := (M'_x)^{\odot b_{xy}} \quad \text{and} \quad V := M_x^{\odot b_{xy}}.$$

Set

$$X := \bigodot_{b_{iy}>0} M_i^{\odot b_{iy}} \simeq M_x^{\odot b_{xy}} \odot P = V \odot P, \quad Y := \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}} \simeq Q \odot A.$$

Then (6.5) reads as

$$(6.6) \quad 0 \longrightarrow q(B \odot P) \longrightarrow q^{\tilde{\Lambda}(M_y, M'_y)} M_y \circ M'_y \longrightarrow L \odot Q \longrightarrow 0.$$

Note that we have

$$(6.7) \quad 0 \rightarrow qC \rightarrow q^{\tilde{\Lambda}(M_x, M'_x)} M_x \circ M'_x \rightarrow M_y^{\odot b_{xy}} \odot S \rightarrow 0 \quad \text{and}$$

$$(6.8) \quad 0 \rightarrow qX \rightarrow q^{\tilde{\Lambda}(M_y, M'_y)} M_y \circ M'_y \rightarrow Y \rightarrow 0.$$

Taking the convolution products of  $L = (M'_x)^{\odot b_{xy}}$  and (6.8), we obtain

$$0 \longrightarrow qL \circ X \longrightarrow q^{\tilde{\Lambda}(M_y, M'_y)} L \circ (M_y \circ M'_y) \longrightarrow L \circ Y \longrightarrow 0,$$

$$0 \longrightarrow qX \circ L \longrightarrow q^{\tilde{\Lambda}(M_y, M'_y)} (M_y \circ M'_y) \circ L \longrightarrow Y \circ L \longrightarrow 0.$$

Since  $L$  commutes with  $M_y$ , we have

$$\Lambda(L, Y) = \Lambda(L, M_y \diamond M'_y)$$

$$= \Lambda(L, M_y) + \Lambda(L, M'_y) = \Lambda(L, M_y \circ M'_y).$$

On the other hand, we have

$$\begin{aligned}
& \Lambda(M'_x, X) - \Lambda(M'_x, Y) \\
&= \Lambda(M'_x, \bigodot_{b_{iy}>0} M_i^{\odot b_{iy}}) - \Lambda(M'_x, \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}}) \\
&= \sum_{b_{iy}>0} \Lambda(M'_x, M_i) b_{iy} - \sum_{b_{iy}<0} \Lambda(M'_x, M_i) (-b_{iy}) \\
&= \sum_{i \in J} \Lambda(M'_x, M_i) b_{iy} = \sum_{i \neq x} \Lambda(M'_x, M_i) b_{iy} + \Lambda(M'_x, M_x) b_{xy} \\
&= \sum_{i \neq x} \Lambda(M'_x, M_i) (b'_{iy} - \delta(b_{ix} > 0) b_{ix} b_{xy}) + \Lambda(M'_x, M_x) b_{xy} \\
&= \sum_{i \neq x} \Lambda(M'_x, M_i) b'_{iy} - \sum_{b_{ix}>0} \Lambda(M'_x, M_i) b_{ix} b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\
&\stackrel{(a)}{=} 0 - \Lambda(M'_x, \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}) b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\
&= (-\Lambda(M'_x, \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}) + \Lambda(M'_x, M_x)) b_{xy} \\
&= (-\Lambda(M'_x, M'_x \diamond M_x) + \Lambda(M'_x, M_x)) b_{xy} \\
&= (-\Lambda(M'_x, M'_x) - \Lambda(M'_x, M_x) + \Lambda(M'_x, M_x)) b_{xy} = 0.
\end{aligned}$$

Note that we used the compatibility of the pair  $((-\Lambda(\mu_x(M_i), \mu_x(M_j)))_{i,j \in J}, \mu_x(\tilde{B}))$  when we derive the equality (a).

Since  $L = (M'_x)^{\odot b_{xy}}$ , the equality  $\Lambda(M'_x, X) = \Lambda(M'_x, Y)$  implies

$$\Lambda(L, X) = \Lambda(L, Y) = \Lambda(L, M_y \circ M'_y).$$

Hence the following diagram is commutative:

$$\begin{array}{ccccccc}
0 & \longrightarrow & qL \circ X & \longrightarrow & q^{\tilde{\Lambda}(M_y, M'_y)} L \circ (M_y \circ M'_y) & \longrightarrow & L \circ Y \longrightarrow 0 \\
(6.9) & & \downarrow \mathbf{r}_{L,X} & & \downarrow \mathbf{r}_{L, M_y \circ M'_y} & & \downarrow \mathbf{r}_{L,Y} \wr \\
0 & \longrightarrow & q^{d+1} X \circ L & \longrightarrow & q^{d+\tilde{\Lambda}(M_y, M'_y)} (M_y \circ M'_y) \circ L & \longrightarrow & q^d Y \circ L \longrightarrow 0,
\end{array}$$

where  $d = -\Lambda(L, X) = -\Lambda(L, M_y \circ M'_y) = -\Lambda(L, Y)$ . Note that since  $L = (M'_x)^{\odot b_{xy}}$  commutes with  $Q$  and  $A$ ,  $\mathbf{r}_{L,Y}$  is an isomorphism and hence we have

$$\text{Im}(\mathbf{r}_{L,Y}) \simeq L \circ Y.$$

Hence we have an exact sequence

$$(6.10) \quad 0 \longrightarrow \text{Im}(\mathbf{r}_{L,X}) \longrightarrow \text{Im}(\mathbf{r}_{L,M_y \circ M'_y}) \longrightarrow L \circ Y \longrightarrow 0.$$

On the other hand,  $\mathbf{r}_{L,M_y \circ M'_y}$  decomposes (up to a grading shift) as follows:

$$\begin{array}{ccccc} & & \mathbf{r}_{L,M_y \circ M'_y} & & \\ & \nearrow & & \searrow & \\ L \circ M_y \circ M'_y & \xrightarrow[\mathbf{r}_{L,M_y \circ M'_y}]{\sim} & M_y \circ L \circ M'_y & \xrightarrow{M_y \circ \mathbf{r}_{L,M'_y}} & M_y \circ M'_y \circ L. \end{array}$$

Since  $L = (M'_x)^{\odot b_{xy}}$  commutes with  $M_y$ , the homomorphisms  $\mathbf{r}_{L,M_y} \circ M'_y$  is an isomorphism and hence we have

$$\text{Im}(\mathbf{r}_{L,M_y \circ M'_y}) \simeq M_y \circ (L \diamond M'_y) \quad \text{up to a grading shift.}$$

Similarly  $\mathbf{r}_{L,X}$  decomposes (up to a grading shift) as follows:

$$\begin{array}{ccccc} & & \mathbf{r}_{L,X} & & \\ & \nearrow & & \searrow & \\ L \circ V \circ P & \xrightarrow[\mathbf{r}_{L,V \circ P}]{\sim} & V \circ L \circ P & \xrightarrow[V \circ \mathbf{r}_{L,P}]{\sim} & V \circ P \circ L. \end{array}$$

Since  $L$  commutes with  $P$ , the homomorphisms  $V \circ \mathbf{r}_{L,P}$  is an isomorphism and hence we have

$$\text{Im}(\mathbf{r}_{L,X}) \simeq (L \diamond V) \circ P \simeq ((M'_x)^{\odot b_{xy}} \diamond M_x^{\odot b_{xy}}) \circ P \quad \text{up to a grading shift.}$$

On the other hand, Lemma 2.23 implies that

$$(M'_x)^{\odot b_{xy}} \diamond M_x^{\odot b_{xy}} \simeq (M'_x \diamond M_x)^{\odot b_{xy}} \simeq C^{\odot b_{xy}} \simeq B \odot A,$$

and hence we obtain

$$\text{Im}(\mathbf{r}_{L,X}) \simeq (B \odot P) \odot A \quad \text{up to a grading shift.}$$

Thus the exact sequence (6.10) becomes the exact sequence in  $\mathcal{C}$ :

$$(6.11) \quad 0 \longrightarrow q^m(B \odot P) \odot A \longrightarrow q^n M_y \circ (L \diamond M'_y) \longrightarrow (L \odot Q) \odot A \longrightarrow 0.$$

for some  $m, n \in \mathbb{Z}$ . Since  $(L \odot Q) \odot A$  is self-dual,  $n = \widetilde{\Lambda}(M_y, L \diamond M'_y)$ . On the other hand, we have

$$\mathfrak{d}(M_y, L \diamond M'_y) \leq \mathfrak{d}(M_y, L) + \mathfrak{d}(M_y, M'_y) = 1.$$

By the exact sequence (6.11),  $M_y \circ (L \diamond M'_y)$  is not simple and we conclude

$$\mathfrak{d}(M_y, L \diamond M'_y) = 1.$$

Then Corollary 2.24 implies that  $m = 1$ . Thus we obtain an exact sequence in  $\mathcal{C}$ :

$$(6.12) \quad 0 \longrightarrow q(B \odot P) \odot A \longrightarrow q^{\tilde{\Lambda}(M_y, L \diamond M'_y)} M_y \odot (L \diamond M'_y) \longrightarrow (L \odot Q) \odot A \longrightarrow 0.$$

Now we shall rewrite (6.12) by using  $\bullet \circ A$  instead of  $\bullet \odot A$ . We have

$$\begin{aligned} \tilde{\Lambda}(B, A) + \tilde{\Lambda}(A, A) &= b_{xy} \tilde{\Lambda}(C, A) = b_{xy} \tilde{\Lambda}(M'_x \diamond M_x, A) \\ &= b_{xy} \tilde{\Lambda}(M'_x, A) + b_{xy} \tilde{\Lambda}(M_x, A) = \tilde{\Lambda}(L, A) + b_{xy} \tilde{\Lambda}(M_x, A). \end{aligned}$$

On the other hand, the exact sequence (6.8) gives

$$\begin{aligned} b_{xy} \tilde{\Lambda}(M_x, A) + \tilde{\Lambda}(P, A) &= \tilde{\Lambda}(X, A) = \tilde{\Lambda}(M'_y \diamond M_y, A) \\ &= \tilde{\Lambda}(M'_y, A) + \tilde{\Lambda}(M_y, A) = \tilde{\Lambda}(M_y \diamond M'_y, A) = \tilde{\Lambda}(Y, A) = \tilde{\Lambda}(Q, A) + \tilde{\Lambda}(A, A). \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{\Lambda}(B \circ P, A) &= \tilde{\Lambda}(B, A) + \tilde{\Lambda}(P, A) \\ &= (\tilde{\Lambda}(L, A) + b_{xy} \tilde{\Lambda}(M_x, A) - \tilde{\Lambda}(A, A)) + (\tilde{\Lambda}(Q, A) + \tilde{\Lambda}(A, A) - b_{xy} \tilde{\Lambda}(M_x, A)) \\ &= \tilde{\Lambda}(L, A) + \tilde{\Lambda}(Q, A) = \tilde{\Lambda}(L \circ Q, A). \end{aligned}$$

Thus we have

$$(6.13) \quad 0 \longrightarrow q(B \odot P) \circ A \longrightarrow q^c M_y \circ (L \diamond M'_y) \longrightarrow (L \odot Q) \circ A \longrightarrow 0,$$

where  $c = \tilde{\Lambda}(M_y, L \diamond M'_y) - \tilde{\Lambda}(B \odot P, A)$ .

Thus we obtain the identity in  $K(R\text{-gmod})$ :

$$q^c [M_y] [L \diamond M'_y] = (q[B \odot P] + [L \odot Q]) [A].$$

On the other hand, hypothesis (6.4) implies that there exists  $\phi \in \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  such that

$$(6.14) \quad [M_y] \phi = q[B \odot P] + [L \odot Q]$$

and

$$(6.15) \quad \phi[\mu_x(M)_i] = q^{\lambda'_{yi}} [\mu_x(M)_i] \phi \quad \text{for } i \neq y,$$

where  $\mu_y \mu_x(-\Lambda) = (\lambda'_{ij})_{i,j \in J}$ .

Hence, in  $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ , we have

$$[M_y] \phi [A] = (q[B \odot P] + [L \odot Q]) [A] = q^c [M_y] [L \diamond M'_y].$$

Since  $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  is a domain, we conclude that

$$\phi[A] = q^c [L \diamond M'_y].$$

On the other hand, (6.15) implies that

$$\phi[A] = q^l [A] \phi \quad \text{for some } l \in \mathbb{Z}.$$

Hence, Theorem 3.3 implies that, when we write

$$\phi = \sum_{b \in B(\infty)} a_b [L_b] \quad \text{for some } a_b \in \mathbb{Q}(q^{1/2}),$$

we have  $L_b \circ A \simeq q^l A \circ L_b$  whenever  $a_b \neq 0$ . Hence we have

$$q^c(L \diamond M'_y) = \phi[A] = \sum_{b \in B(\infty)} a_b [L_b \circ A].$$

We conclude that there exists a self-dual simple module  $M''_y$  in  $R\text{-gmod}$  such that  $M''_y$  commutes with  $A$  and

$$\phi = q^m [M''_y]$$

for some  $m \in \mathbb{Z}$ . Then (6.14) implies that

$$q^m [M_y \circ M''_y] = q[B \odot P] + [L \odot Q].$$

Hence there exists an exact sequence

$$0 \longrightarrow X \longrightarrow q^m M_y \circ M''_y \longrightarrow Y \longrightarrow 0,$$

where  $X = qB \odot P$  and  $Y = L \odot Q$  or  $X = L \odot Q$  and  $Y = qB \odot P$ . By Corollary 2.24, the last case does not occur and we have an exact sequence

$$0 \longrightarrow qB \odot P \longrightarrow q^m M_y \circ M''_y \longrightarrow L \odot Q \longrightarrow 0.$$

Since  $M_y$ ,  $M''_y$  and  $L \odot Q$  are self-dual, we have  $m = \tilde{\Lambda}(M_y, M''_y)$ , and we obtain the desired short exact sequence (6.6).

Since  $\phi$  commutes with  $[\mu_x(M)_i]$  up to a power of  $q$  in  $K(\mathcal{C})$ , and  $\mu_x(M)_i$  is real simple,  $M''_y$  commutes with  $\mu_x(M)_i$  for  $i \neq y$ .  $\square$

**Corollary 6.4.** *Let  $(\{M_i\}_{i \in J}, \tilde{B})$  be an admissible pair in  $\mathcal{C}$ . Under the assumption (6.4),  $\mathcal{C}$  is a monoidal categorification of the quantum cluster algebra  $\mathcal{A}_{q^{1/2}}([\mathcal{S}])$ . Furthermore, we have the followings:*

- (i) *The quantum monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in J}, -\Lambda, \tilde{B}, D)$  admits successive mutations in all directions.*
- (ii) *Any cluster monomial in  $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  is the isomorphism class of a real simple object in  $\mathcal{C}$  up to a power of  $q^{1/2}$ .*
- (iii) *Any cluster monomial in  $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$  is a Laurent polynomial of the initial cluster variables with coefficient in  $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$ .*

*Proof.* (i) and (ii) are straightforward.

Let us show (iii). Let  $x$  be a cluster monomial. By the Laurent phenomenon ([2]), we can write

$$xX^{\mathbf{c}} = \sum_{\mathbf{a} \in \mathbb{Z}_{\geq 0}^J} c_{\mathbf{a}} X^{\mathbf{a}},$$

where  $X = (X_i)_{i \in J}$  is the initial cluster,  $\mathbf{c} \in \mathbb{Z}_{\geq 0}^J$  and  $c_{\mathbf{a}} \in \mathbb{Q}(q^{\pm 1/2})$ . Since  $x$  and  $X^{\mathbf{c}}$  are the isomorphism classes of simple modules up to a power of  $q^{1/2}$ , their product  $xX^{\mathbf{c}}$  can be written as a linear combination of the isomorphism classes of simple modules with coefficients in  $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$ . Since every  $X^{\mathbf{a}}$  is the isomorphism class of a simple module up to a power of  $q^{1/2}$ , we have  $c_{\mathbf{a}} \in \mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$ .  $\square$

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